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NUMERICAL COMPUTATION OF FITZHUGH-NAGUMO EQUATION: A NOVEL GALERKIN FINITE ELEMENT APPROACH

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ABSTRACT

The key objective of this research paper is to find the numerical solution of the famous FitzHugh-Nagumo equation. The numerical scheme used here is the Galerkin finite element method (GFEM) in a simple and convenient way. Because the advantages of using GFEM are that it can be used directly without any linearization or any other restrictive assumption, it uses shape functions instead of trial functions, and it gives a polynomial at each point instead of value, so it can be used to find value at any point within the domain. First we derive the detail formulation of GFEM for this nonlinear parabolic partial differential equation. Then we solve the FitzHugh-Nagumo equation for various values of. Later, we solve another renowned Newell-Whitehead equation for the verification of the consistency of this algorithm. The results are depicted both graphically and numerically. All results are compared with the analytical solutions to show the convergence of the proposed algorithm. Those results demonstrate that our proposed algorithm works efficiently and gives a very good agreement with the exact solution. This can be applied for solving any nonlinear parabolic partial differential equation (PDE).

Contribution/Originality: The study uses the new estimation methodology for approximating the numerical solution of the well-known FitzHug-Nagumo equation by GFEM. It is also remarked that this study is one of very few studies which have approximated the famous F-N partial differential equation using a new technique.

1. INTRODUCTION

In recent years, FitzHugh-Nagumo (F-N) equation proposed by Hodgkin and Huxley [1] has been attracted by a considerable amount of researchers due its importance in various fields of science; instantly see the references [2-9] for example and the branching areas of application:

- Application to neuron dynamics and neurophysiology, an excitable media.
- Flame propagation.
- Logistic population growth.
- Branching Brownian motion process and circuit theory.
- The transmission of nerve impulses.
- The area of population genetics.
- Autocatalytic chemical reaction and nuclear reactor theory.

It is a nonlinear parabolic partial differential equation generally expressed as:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u(u-\lambda)(1-u) \tag{1}$$

Where λ is an arbitrary constant and $0\leq\lambda\leq1$. When $\lambda=1$ Equation 1 reduces to the following equation.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u-u^3 \tag{2}$$

Equation 2 is the famous real Newell-Whitehead equation. The F-N Equation 1 has been studied by many mathematicians and physicists. In Shih, et al. [10] Shih applied approximate conditional symmetries method to obtain first-order approximate solutions of the perturbed FitzHugh-Nagumo equation. Abbasbandy [4] applied an analytic technique, the homotopy analysis approach to obtain the soliton solution of the F-N equation. Bhrawy [11] used Jacobi–Gauss–Lobatto collocation method for solving the generalized FitzHugh–Nagumo equation. The Haar wavelet method has been presented by Hariharan and Kannan [12] for solving the standard FitzHugh–Nagumo equation. Chen, et al. [13] presented semi-explicit finite-difference method for generalized Nagumo reaction-diffusion equation where Feng and Lin [14] used finite difference method to approximate traveling wave solutions of the FitzHugh-Nagumo equations. Van Gorder [15] exercised the variational method to obtain analytical solutions for both the Nagumo telegraph and the Nagumo reaction–diffusion partial differential equations. In Teodoro [16] Teodoro used finite element method for solving FitzHugh-Nagumo equation but the application was in linearized equation that was converted from nonlinear to linear by applying Newton method.

In the literature and to the best of our knowledge, still none have attempted to solve the famous FitzHugh-Nagumo equation by GFEM directly. The main novelty of this paper is that we apply Galerkin finite element method for the numerical solution of FitzHugh-Nagumo equation directly without linearization of the nonlinear terms.

Now it is time to discuss about the mathematical formulation of Galerkin finite element algorithm to solve FitzHugh-Nagumo equation.

2. FITZHUGH-NAGUMO EQUATION AND GALERKIN FEM

Galerkin Finite Element Method (GFEM) is an efficient numerical method for approximating numerical solutions of various problems in different branches of science. In this section, we formulate GFEM for the numerical solution of FitzHugh-Nagumo equation given as:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u(u-\lambda)(1-u); \qquad (x,t)\in[A,B]\times[0,T] \tag{3}$$

Subject to the boundary conditions

$$\frac{\partial u(x,t)}{\partial x}\bigg|_{x=A}=\Gamma_1(t); \qquad t\in[0,T] \tag{4a}$$

$$\frac{\partial u(x,t)}{\partial x}\bigg|_{x=B}=\Gamma_2(t) \qquad t\in[0,T] \tag{4b}$$

Equations 4a and 4b represent the left and right boundary conditions, respectively of Equation 3. And Equation 5 represents the initial condition which is the function of spatial variable x only.

$$u(x,t)|_{t=0}=\Lambda(x) \;; \quad x\in[A,B] \tag{5}$$

For applying GFEM, we use linear shape functions

$$\Phi_1=\frac{1-\varsigma}{2} \qquad \varsigma\in[-1,1]$$

$$\Phi_2 = \frac{1+\varsigma}{2} \qquad \varsigma \in [-1,1]$$

In parabolic partial differential equations, linear shape functions give better accuracy than quadratic and other shape functions.

Then the trial solution for Equation 1 will be:

$$\tilde{u}(\varsigma, t) = \sum_{j=1}^{n} u_j(t)\Phi_j(\varsigma) \tag{6}$$

To apply GFEM, first of all we have to discretize the domain $[A, B]$ of ς into n sub-interval. Let the space increment be h and the time increment be k. Each sub-interval of ς is called an *element*. Finite element method generates a polynomial for each sub-interval.

The weighted residual equation of the F-N equation after applying trial solution for a particular element $[e]$ becomes

$$\int_{[e]} \left(\frac{\partial \tilde{u}}{\partial t} - \frac{\partial^2 \tilde{u}}{\partial x^2} - \tilde{u}(\tilde{u} - \lambda)(1 - \tilde{u}) \right) \Phi_i dx = 0$$

which yields

$$\int_{[e]} \frac{\partial \tilde{u}}{\partial t} \Phi_i dx - \int_{[e]} \frac{\partial \tilde{u}}{\partial x} \frac{\partial \Phi_i}{\partial x} dx - \int_{[e]} (\tilde{u} - \lambda)(1 - \tilde{u})\Phi_i dx = \left[\frac{\partial \tilde{u}}{\partial x} \Phi_i \right]_{[e]}$$

After using Equation 6 and simple calculation, we obtain

$$\sum_{j=1}^{n} \frac{\partial \tilde{u}}{\partial t} \int_{[e]} \Phi_i \, \Phi_j dx + \sum_{j=1}^{n} u_j \int_{[e]} \frac{\partial \Phi_i}{\partial x} \frac{\partial \Phi_j}{\partial x} dx$$
$$+ \sum_{j=1}^{n} u_j \int_{[e]} \left[\left(\sum_{j=1}^{n} u_j \Phi_j \right)^2 - (1+\lambda) \left(\sum_{j=1}^{n} u_j \Phi_j \right) + \lambda \right] \Phi_i \Phi_j dx$$
$$= \left[\sum_{j=1}^{n} u_j \frac{\partial \Phi_j}{\partial x} \Phi_i \right]_{[e]}$$

The convenient matrix form is

$$\mathbb{C} \frac{du}{dt} + \mathbb{H} = \mathbb{F} \tag{7}$$

Where the entries of the matrices $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{F}$ are expressed in the following Equations 8, 9 and 10, respectively.

$$\mathbb{C}_{ij} = \int_{[e]} \Phi_i \Phi_j dx \tag{8}$$

$$\mathbb{H}_{ij} = \int_{[e]} \left[\frac{\partial \Phi_i}{\partial x} \frac{\partial \Phi_j}{\partial x} + \left\{ \left(\sum_{j=1}^{n} u_j \Phi_j \right)^2 - (1+\lambda) \left(\sum_{j=1}^{n} u_j \Phi_j \right) + \lambda \right\} \Phi_i \Phi_j \right] dx \tag{9}$$

$$\mathbb{F}_{ij} = \left[\sum_{j=1}^{n} u_j \frac{\partial \Phi_j}{\partial x} \Phi_i \right]_{[e]} \tag{10}$$

Equation 7 is a first order ordinary differential equation (ODE) with t as an independent variable. For solve our problem, we have to transform this equation into an algebraic equation. We can do this by various numerical schemes such as Forward difference scheme, Backward difference scheme, Central difference scheme, Crank-Nicolson scheme, Galerkin method etc. Every method has advantages and disadvantages based on the problem under study. The forward and backward difference algorithms are conditionally stable with order of accuracy $O(\Delta t)$ while the Crank-Nicolson and the Galerkin method are unconditionally stable with order of accuracy $O(\Delta t)^2$. In this paper, we use α family of approximation that combines all the schemes discussed above for different values of α [17]. Then the ODE (7) becomes:

$$\left(\frac{\mathbb{C}}{k} + \mathbb{H} \right) u_{j+1} = \left(\frac{\mathbb{C}}{k} + (1-\alpha)\mathrm{H} \right) + (1-\alpha)\mathrm{F}_j + \alpha \mathrm{F}_{j+1} \tag{11}$$

By using the initial and boundary conditions to recurrent Equation 11, we will find the approximate solution of the FitzHugh-Nagumo equation. In the following section we will demonstrate the efficiency of this method by solving the equation for various parametric values of λ and c.

3. APPLICATIONS AND RESULT DISCUSSION

In this section, we apply GFEM on FitzHugh-Nagumo equation according to the algorithm discussed in the previous section. We will present our results both graphically in diagram and numerically via tabular form. In each format, we will compare our approximate results with the exact solution.

In the following example, let us consider the FitzHugh-Nagumo Equation 3 with boundary conditions:

$$\Gamma_1(t) = \frac{1}{4\sqrt{2}} csch^2 \left(-\frac{1}{2\sqrt{2}} + \frac{t}{4} + c \right) \tag{12a}$$

$$\Gamma_2(t) = \frac{1}{4\sqrt{2}} csch^2 \left(\frac{1}{2\sqrt{2}} + \frac{t}{4} + c \right) \tag{12b}$$

The boundary conditions given in Equation's 12a and 12b are the non-homogeneous *Neumann boundary condition.*

And the initial condition is:

$$\Lambda(x) = \frac{1}{2}\left[1 - coth \left(\frac{x}{2\sqrt{2}} + c \right) \right]$$

The exact solution [9] of Equation 3 is:

$$u(x,t) = \frac{1}{2}\left[1 - \coth \left(\frac{x}{2\sqrt{2}} + \frac{2\lambda - 1}{4} t + c \right) \right]$$

For numerical computation, we consider $c = \pi / 4$ and $\lambda = 1, -1$ and $3/4$. Here in GFEM, we take 41 nodes and the shape functions are linear. For shifting Equation 7 into Equation 11, we use α family of approximation with $\alpha = 0.4$ and for iterative purpose, we recall *Picard iterative method* [17].

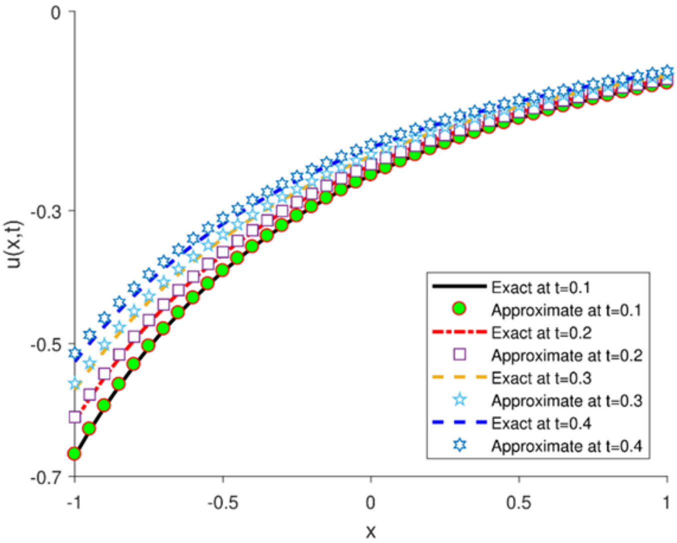
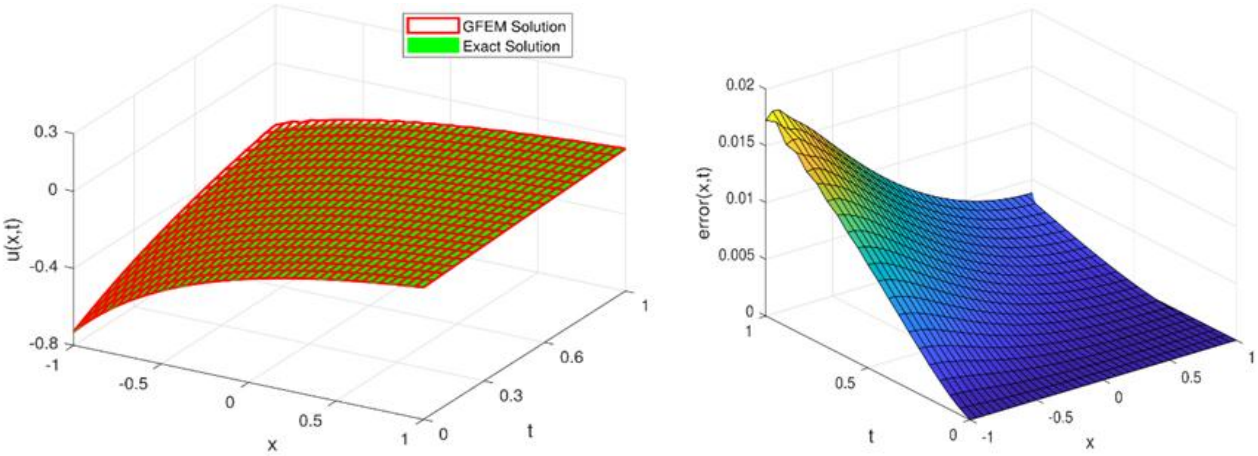


Figure-1. Numerical vs analytical solutions of FitzHugh-Nagumo equation at $\lambda = 1$.



(a) Numerical and exact solutions of (3) at $\lambda = 1$

(b) The absolute error map for GFEM at $\lambda = 1$

Figure-2. Comparative solutions of (3) and error analysis for $\lambda = 1$

From Table 1 it is seen that our proposed method gives a good accuracy for different time circles. Figure 1 depicted the approximate and the analytical solutions for various time *t*. It is noted that for drawing Figures and computing table, we have used MATLAB R2017a code. From this diagram, visibly it is observed that the numerical solution almost coincides with the exact solution.

Table-1. Comparison between approximate and exact solution of FitzHugh-Nagumo equation when $\lambda=1$, $c=\pi/4$ and $\alpha=0.4$

X	t=0.05			t=0.1			t=0.3		
	Exact	GFEM	Error	Exact	GFEM	Error	Exact	GFEM	Error
-1.0	-0.69712	-0.69835	1.14387×10^{-03}	-0.66689	-0.66957	2.50674×10^{-03}	-0.56025	-0.56956	9.02054×10^{-03}
-0.8	-0.55415	-0.55513	8.85114×10^{-04}	-0.53179	-0.53410	2.14197×10^{-03}	-0.45141	-0.45990	8.16902×10^{-03}
-0.6	-0.44837	-0.44905	5.67613×10^{-04}	-0.43150	-0.43316	1.51024×10^{-03}	-0.36962	-0.37642	6.52572×10^{-03}
-0.4	-0.36765	-0.36803	3.55468×10^{-04}	-0.35461	-0.35572	9.90636×10^{-04}	-0.30618	-0.31133	4.91122×10^{-03}
-0.2	-0.30444	-0.30471	2.26779×10^{-04}	-0.29426	-0.29497	6.40413×10^{-04}	-0.25588	-0.25961	3.56162×10^{-03}
0.0	-0.25414	-0.25431	1.48977×10^{-04}	-0.24602	-0.24648	4.18830×10^{-04}	-0.21527	-0.21791	2.52647×10^{-03}
0.2	-0.21349	-0.21361	1.00810×10^{-04}	-0.20694	-0.20724	2.80000×10^{-04}	-0.18205	-0.18389	1.77489×10^{-03}
0.4	-0.18028	-0.18036	7.01071×10^{-05}	-0.17493	-0.17512	1.92107×10^{-04}	-0.15457	-0.15586	1.25298×10^{-03}
0.6	-0.15288	-0.15293	5.01114×10^{-05}	-0.14846	-0.14860	1.36245×10^{-04}	-0.13165	-0.13258	9.11052×10^{-04}
0.8	-0.13010	-0.13014	3.74355×10^{-05}	-0.12641	-0.12652	1.03060×10^{-04}	-0.11238	-0.11312	7.14287×10^{-04}
1.0	-0.11103	-0.11107	3.21607×10^{-05}	-0.10793	-0.10803	9.12198×10^{-05}	-0.09607	-0.09676	6.48389×10^{-04}

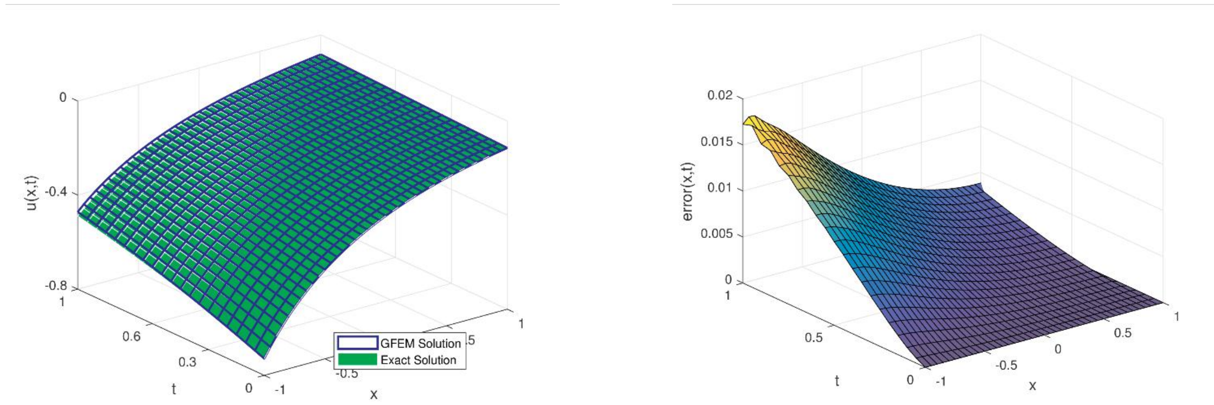
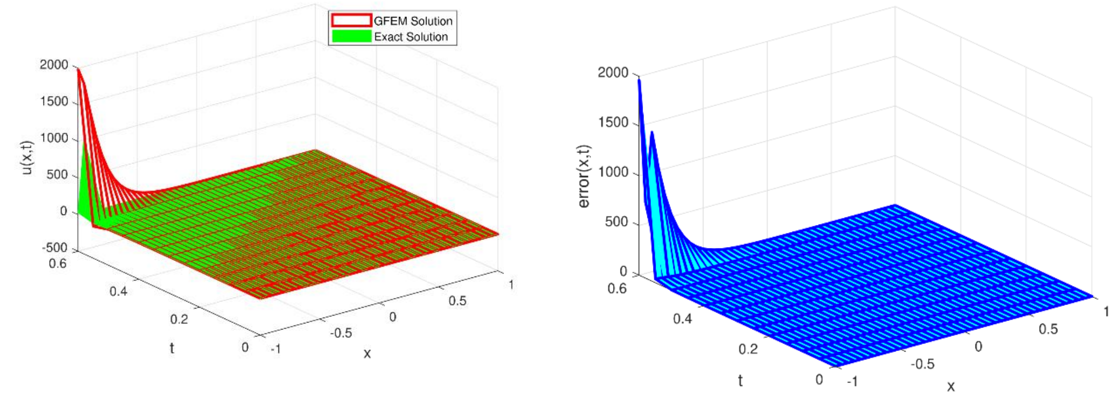


Figure-3. Comparative solutions of (3) and error analysis for $\lambda=3/4$.



(a) Numerical and exact solutions of (3) at $\lambda=-1$.

(b) The absolute error map for GFEM at $\lambda=-1$.

Figure 4. Comparative solutions of (3) and error analysis for $\lambda=-1$.

In Figure 2a, we show the three-dimensional view of the comparison between the exact and approximate solutions. But for the good agreement of the approximate solution to the exact solution, it is hard to distinguish. That is why we use a transparent graph for the approximate solution curve. For better understanding we also provide the error map over time and space domain in Figure 2b which shows a very good similarity with the exact solution, i.e. eventually the error tends to zero.

We also evaluate the solutions of FitzHugh-Nagumo equation for $\lambda = 3/4$ and the results are graphically depicted in Figure 3. On 3D surface, the exact and analytic solutions are matching a good agreement with each other and inherently the error is still feasible.

Finally, we consider the parametric value of λ as $\lambda = -1$ and the FitzHugh-Nagumo equation turn to another famous equation known as Newell Whitehead (N-W) equation. This is a new case study and as far we know from the literature that no one solve it numerically using Galerkin finite element method; even using any numerical methods. Perhaps the main reason is that there is an asymptote in its domain while solving the N-W equation. In this study, we also solve this equation and the results of Newell Whitehead equation is presented in Figure 4; for details see in Figure 4a and Figure 4b.

4. CONCLUSION

In this paper, we derived the complete formulation of Galerkin finite element method for FitzHugh-Nagumo equation. We have solved a particular FitzHugh-Nagumo equation using the proposed algorithm. The results are presented in a data structured table and sketching various graphical maps. By observing all those figures and table, it is clear that the presented outcome exhibits the higher estimated order of convergence of this method. So, we can conclude that GFEM is an efficient, unconditionally stable, highly modular and easily expandable method that can be applied for the solution of more complicated engineering problems.

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