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UNSTEADY PLANE COUETTE FLOW OF AN INCOMPRESSIBLE COUPLE STRESS FLUID WITH SLIP BOUNDARY CONDITIONS

Hikmat S. Saad[1]† --- **E. A. Ashmawy**[2]
[1,2]*Department of Mathematics and Computer Science, Faculty of Science, Beirut Arab University, Beirut, Lebanon*

ABSTRACT

In this work, the unsteady flow of an incompressible couple stress fluid between two parallel plates is studied. Slip boundary conditions are applied on the two plates and vanishing couple stress condition at the boundaries is assumed. The upper plate is suddenly moved with time dependent velocity while the other plate is fixed. The problem is solved analytically in the Laplace domain through the use of Laplace transform technique. The inverse transform of the fluid velocity is obtained numerically. The velocity profiles for different times and different physical parameters are plotted and the numerical results are discussed.

Keywords: Couple stresses fluids, Incompressible, Couette flow, Unsteady flow, Slip boundary condition, Laplace transform.

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Contribution/ Originality

This paper contributes to the field of couple stress fluids by applying the more realistic slip condition to Couette flow of an incompressible couple stress fluid. It is found that the time, the couple stress coefficient and the slip parameters have significant effect on the flow field.

1. INTRODUCTION

After many studies acknowledged the inadequacy of the classical Navier-Stokes theory for describing rheological complex fluids, researchers attempt to develop several theories related to non-Newtonian fluids. The studies of non-Newtonian fluids have gained great momentum recently because of the belief of various researchers that such theories can have widespread contributions and applications in an array of scientific, technological, and industrial fields. One new branch of the non-Newtonian fluids which has witnessed a growing interest in the last five decades is namely the couple stress fluid.

The couple stress fluid theory, introduced first in 1966 by Stokes [1] is one kind of the various polar fluid theories which consider not only the classical Cauchy stress but also couple stresses. Couple stress fluids are fluids that contain rigid randomly oriented particles suspended in a viscous medium. Stokes developed this theory which preserves the presence of couple stresses and body couples and represents the simplest generalization of the classical fluid theory. An excellent introduction to the couple stress fluid theory is described elaborately by Stokes [2] himself in his treatise. The theory entirely sheds lights on the potential effects of couple stresses assuming that the fluid has no microstructure at the kinematical level, and it's the velocity field that determines the kinematics of motion. What characterizes the concept of couple stresses is the way in which the mechanical interactions in the fluid medium are represented and the stress tensor is not symmetric. The essential equations associated with the couple stress fluid flow are aligned by those of the classical Navier-Stokes equations, yet the order of the differential equations in the couple stress fluid is increased by two.

† Corresponding author

This couple stress theory has been broadly used due to its relative mathematical simplicity in comparison to the previous models developed for the polar fluids. Numerous applications have been recorded in the theory of lubrication by Naduvinamani, et al. [3] in addition to modeling the flow of synthetic fluid, polymer thickened oils, liquid crystals, animal blood, and synovial fluid present in synovial joints [4, 5].

The Navier-Stokes equations of fluid flow are typically solved assuming the validity of the no-slip boundary conditions where all the three components of the fluid velocity on a solid surface are equal to the respective velocity components of the surface. The no-slip boundary condition, applicable when a viscous fluid flows over a solid surface, was an inevitable consequence in the early development of fluid mechanics. Under normal conditions, the no slip boundary appeared to provide realistic explanations and experimental evidence for those who incorporated such an assumption. However, in the studies that were conducted during the previous century, this assumption has become much less clear and it was shown that such condition might not always hold. In fact, going back to 1823, Navier [6] proposed a slip boundary condition which suggests that the tangential velocity of fluid relative to the solid boundary at a point on its surface is proportional to the tangential stress acting at that point. Neto, et al. [7] have made a review of experimental studies on the boundary slip of Newtonian liquids. In view of this, several researchers have studied different flow problems in various configurations while using the slip boundary condition. Lately, Ashmawy [8] studied the unsteady plane Couette flow of micropolar fluid with slip condition. Neill, et al. [9] used the Basset type linear slip boundary conditions to remove the contact-line singularity. More recent studies and researches discussing the effect of slip conditions in viscous and micropolar fluid flows were conducted. Ellahi [10] studied the effect of the slip boundary condition on non-Newtonian flows in a channel, Yang and Zhu [11] investigated the analytical solution for squeeze flow of the Bingham fluid with slip, Sherief, et al. [12] discussed the slip flow between two confocal rotating spheroids. The fully developed natural convective micropolar fluid flow in a vertical channel with slip and the unsteady slip flow of a micropolar fluid between parallel plates were solved by [13, 14].

Devakar and Iyengar [15] obtained the solution of the flow of couple stress fluid between concentric cylinders with slip boundary conditions. These same authors have already conducted a thorough investigation of the flow of incompressible couple stress fluid between two parallel plates using the slip boundary condition [16]. Recently, Shantha, et al. [17] studied the unsteady Plane Couette flow of a couple stress fluids with no slip conditions. To the best knowledge of the author, the study of the unsteady plane Couette flow of an incompressible couple stress fluid have not been solved subject to slip boundary condition.

The focus of this work is the unsteady couple stress fluid flow between two infinite horizontal parallel plates distant 2h apart, where the upper plate is moving with a time dependent velocity and the lower plate is kept stationary. The effects of time, slip and couple stress parameters are studied. The Laplace transform technique and differential equation methods are used to obtain the analytical expression of the velocity in the Laplace domain. A standard numerical inversion technique is used to invert the Laplace transform of the velocity. The numerical results are presented graphically and discussed.

2. GOVERNING EQUATIONS, PROBLEM FORMULATION AND SOLUTION

The flow of an incompressible couple stress fluid, in the absence of body forces and body couples, is described by the following differential equations:

$$\nabla \cdot \vec{q} = 0 \tag{1}$$

$$\rho \frac{d\vec{q}}{dt} = -\nabla p - \mu \nabla \times \nabla \times \vec{q} - \eta \nabla \times \nabla \times \nabla \times \nabla \times \vec{q}, \tag{2}$$

Where ρ and $\vec{q}$ are the density and the velocity of the fluid respectively, p is the fluid pressure at any point, μ denotes the viscosity coefficient and η is the couple stress viscosity coefficient.

The constitutive equation connecting the force stress tensor and the rate of deformation tensor is given by:

$$t_{ij} = -p\,\delta_{ij} + 2\mu d_{ij} - \frac{1}{2}\varepsilon_{ijk}\left(m_{,k} + 4\eta\, w_{k,rr}\right), \tag{3}$$

Where the deformation tensor is given by

$$d_{ij} = \frac{1}{2}\left(q_{i,j} + q_{j,i}\right) \tag{4}$$

The couple stress tensor is represented as follow

$$m_{ij} = \frac{1}{3} m\delta_{ij} + 4\eta\, w_{i,j} + 4\eta'\, w_{j,i} \ , \tag{5}$$

Where $\vec{w} = \frac{1}{2}\nabla \times \vec{q}$ is the spin vector, $w_{i,j}$ is the spin tensor, m is the trace of the couple stress tensor m_{ij} and η' is the couple stress viscosity coefficient. δ_{ij}, ε_{ijk} represent respectively the Kronecker delta and the alternating tensor.

Note that, as $\eta \to 0$, equation (2) reduces to the Navier-Stokes equation of motion for classical viscous fluid. Also, the viscosity parameters satisfy the inequalities $\mu \geq 0,\ \ \eta \geq 0,\ \ |\eta'| \leq \eta$.

Consider the unsteady flow of an incompressible couple stress fluid between two infinite horizontal plates separated by a distance $2h$. Assume that the two plates are initially at rest. At time $t = 0^+$, the upper plate is suddenly moved with an arbitrary velocity $U = v_0 f(t)$, setting the fluid into motion, where v_0 is a constant with dimensions of velocity, while $f(t)$ is an arbitrary function of time t.

This is the case of an unsteady unidirectional flow described in Cartesian coordinate (x, y, z), where the x-axis is parallel to the flow direction, the origin of the Cartesian system is on the plane of symmetry of the flow, the y-axis is normal to the plates and the z-axis is perpendicular to the plane of the flow.

For unsteady unidirectional flow, the only non-zero component of the velocity of the fluid is the x-component, the velocity field is $\vec{q} = (u(y,t),0,0)$, which satisfies the equation of continuity (1) automatically.
While the equation of conservation of momentum (2) reduces to

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \eta \frac{\partial^4 u}{\partial y^4}\ . \tag{6}$$

The initial and the slip boundary conditions applied to the problem are assumed to be of the form

$$u(y,0) = 0 \quad \text{for all} \quad y, \tag{7}$$

$$\beta_1\left(u(h,t) - v_0 f(t)\right) = \left(-\mu \frac{\partial u}{\partial y} + \eta \frac{\partial^3 u}{\partial y^3}\right)_{y=h}, \tag{8}$$

$$\beta_2\left(u(-h,t) - 0\right) = \left(\mu \frac{\partial u}{\partial y} - \eta \frac{\partial^3 u}{\partial y^3}\right)_{y=-h}, \tag{9}$$

β_1 *and* β_2 are respectively the slip parameters of the upper and the lower plate such that $0 \leq \beta_1, \beta_2 \leq \infty$. These parameters depend only on the material of the plates and on the nature of the fluid. As a special case, when $\beta_1, \beta_2 \to \infty$ we obtain the classical no-slip case.

The boundary conditions of vanishing couple stresses are of the form

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=h} = 0\,, \tag{10}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=-h} = 0 . \tag{11}$$

Now, we introduce the following non-dimensional variables

$$\hat{u} = \frac{u}{v_0}, \quad \hat{y} = \frac{y}{h}, \quad \hat{t} = \frac{v_0}{h} t .$$

In view of these variables, equation (6) can be written in the form

$$\frac{\partial^4 \hat{u}}{\partial \hat{y}^4} - a^2 \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + R \frac{\partial \hat{u}}{\partial \hat{t}} = 0 , \tag{12}$$

Where $R = \rho v_0 h^3 / \eta$ and $a = \sqrt{\mu h^2 / \eta}$.

After dropping the hats, equation (12) becomes

$$\frac{\partial^4 u}{\partial y^4} - a^2 \frac{\partial^2 u}{\partial y^2} + R \frac{\partial u}{\partial t} = 0 \cdot \tag{13}$$

In terms of the non-dimensional variables, equations (8) and (9) are reduced, after dropping the hats for convenience

$$u(1,t) - f(t) = \left(-\frac{1}{\alpha_1} \frac{\partial u}{\partial y} + \frac{1}{\alpha_1 a^2} \frac{\partial^3 u}{\partial y^3} \right)_{y=1} , \tag{14}$$

$$u(-1,t) = \left(\frac{1}{\alpha_2} \frac{\partial u}{\partial y} - \frac{1}{\alpha_2 a^2} \frac{\partial^3 u}{\partial y^3} \right)_{y=-1} , \tag{15}$$

Such that $\alpha_i = h\beta_i / \mu , \; i = 1, 2$.

Also, equations (7), (10) and (11) become after applying the non-dimensional variables and dropping the hats

$$u(y,0) = 0 \quad for\, all \;\; y , \tag{16}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=1} = 0 , \tag{17}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=-1} = 0 . \tag{18}$$

Introducing the Laplace transform defined by the formula

$$\overline{F}(y,s) = \int_0^\infty e^{-st} F(y,t) dt ,$$

The partial differential equation (13) can be reduced to

$$\frac{\partial^4 \overline{u}}{\partial y^4} - a^2 \frac{\partial^2 \overline{u}}{\partial y^2} + Rs\overline{u} = 0 . \tag{19}$$

The vanishing couple stresses and slip boundary conditions (14), (15), (17) and (18), after applying the Laplace transform, take the form

$$\overline{u}(1,s) - \overline{f}(S) = \left(-\frac{1}{\alpha_1} \frac{\partial \overline{u}}{\partial y} + \frac{1}{\alpha_1 a^2} \frac{\partial^3 \overline{u}}{\partial y^3} \right)_{y=1} , \tag{20}$$

$$\overline{u}(-1,s) = \left(\frac{1}{\alpha_2} \frac{\partial \overline{u}}{\partial y} - \frac{1}{\alpha_2 a^2} \frac{\partial^3 \overline{u}}{\partial y^3} \right)_{y=-1} , \tag{21}$$

$$\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)_{y=1} = 0 \ , \tag{22}$$

$$\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)_{y=-1} = 0 \ . \tag{23}$$

The solution of equation (19), in the Laplace domain, is given by

$$\bar{u}(y,s) = A_1 e^{m_1 y} + A_2 e^{-m_1 y} + A_3 e^{m_2 y} + A_4 e^{-m_2 y} \ , \tag{24}$$

Where m_1 and m_2 are the roots of the characteristic equation

$m^4 - a^2 m^2 + Rs = 0$.

Substituting equation (24) into equations (20), (21), (22) and (23), we get a system of four equations in four unknowns A_1, A_2, A_3 and A_4.

The expressions of A_1, A_2, A_3 and A_4 as follows

$$A_1 = [\xi_{14}(\xi_3\xi_8 - \xi_4\xi_7) + \xi_{15}(\xi_4\xi_6 - \xi_2\xi_8) + \xi_{16}(\xi_2\xi_7 - \xi_3\xi_6)] f(s)/D,$$
$$A_2 = [\xi_1(\xi_{15}\xi_8 - \xi_{16}\xi_7) + \xi_{13}(\xi_4\xi_7 - \xi_3\xi_8) - \xi_5(\xi_{15}\xi_4 - \xi_{16}\xi_3)] f(s)/D,$$
$$A_3 = [\xi_1(\xi_{14}\xi_8 - \xi_{16}\xi_6) + \xi_{13}(\xi_4\xi_6 - \xi_2\xi_8) - \xi_5(\xi_{15}\xi_4 - \xi_{16}\xi_2)] f(s)/D,$$
$$A_4 = [\xi_1(\xi_{14}\xi_7 - \xi_{15}\xi_6) + \xi_{13}(\xi_3\xi_6 - \xi_2\xi_7) - \xi_5(\xi_{14}\xi_3 - \xi_{15}\xi_2)] f(s)/D,$$

With

$$\begin{aligned} D = {} & \xi_1[\xi_{10}(\xi_{15}\xi_8 - \xi_{16}\xi_7) + \xi_{11}(\xi_{16}\xi_6 - \xi_{14}\xi_8) + \xi_{12}(\xi_{14}\xi_7 - \xi_{15}\xi_6)] - \xi_{10}[\xi_{13}(\xi_3\xi_8 - \xi_4\xi_7) + \xi_5(\xi_{15}\xi_4 - \xi_{16}\xi_3)] \\ & + \xi_{11}[\xi_{13}(\xi_2\xi_8 - \xi_4\xi_6) + \xi_5(\xi_{14}\xi_4 - \xi_{16}\xi_2)] - \xi_{12}[\xi_{13}(\xi_2\xi_7 - \xi_3\xi_6) + \xi_5(\xi_{14}\xi_3 - \xi_{15}\xi_2)] \\ & + \xi_9[\xi_{14}(\xi_3\xi_8 - \xi_4\xi_7) + \xi_{15}(\xi_4\xi_6 - \xi_2\xi_8) + \xi_{16}(\xi_2\xi_7 - \xi_3\xi_6)] \end{aligned}$$

$$\xi_1 = m_1^2 e^{m_1} = \xi_6, \quad \xi_2 = m_1^2 e^{-m_1} = \xi_5, \quad \xi_3 = m_2^2 e^{m_2} = \xi_8, \quad \xi_4 = m_2^2 e^{-m_2} = \xi_7$$

$$\xi_9 = e^{m_1} + \frac{m_1}{\alpha_1} e^{m_1} - \frac{m_1^3}{\alpha_1 a^2} e^{m_1}, \quad \xi_{10} = e^{-m_1} - \frac{m_1}{\alpha_1} e^{-m_1} + \frac{m_1^3}{\alpha_1 a^2} e^{-m_1}$$

$$\xi_{11} = e^{m_2} + \frac{m_2}{\alpha_1} e^{m_2} - \frac{m_2^3}{\alpha_1 a^2} e^{m_2}, \quad \xi_{12} = e^{-m_2} - \frac{m_2}{\alpha_1} e^{-m_2} + \frac{m_2^3}{\alpha_1 a^2} e^{-m_2}$$

$$\xi_{13} = e^{-m_1} - \frac{m_1}{\alpha_2} e^{-m_1} + \frac{m_1^3}{\alpha_2 a^2} e^{-m_1}, \quad \xi_{14} = e^{m_1} + \frac{m_1}{\alpha_2} e^{m_1} - \frac{m_1^3}{\alpha_2 a^2} e^{m_1}$$

$$\xi_{15} = e^{-m_2} - \frac{m_2}{\alpha_2} e^{-m_2} + \frac{m_2^3}{\alpha_2 a^2} e^{-m_2}, \quad \xi_{16} = e^{m_2} + \frac{m_2}{\alpha_2} e^{m_2} - \frac{m_2^3}{\alpha_2 a^2} e^{m_2}$$

3. THE NUMERICAL INVERSION OF LAPLACE TRANSFORM

By adopting a numerical inversion technique developed by Honig and Hirdes [18] the expression of the velocity $u(y,t)$ is obtained. An outline of this method is mentioned below.

The inverse Laplace transform of the function $\bar{g}(s)$ denoted by $g(t) = L^{-1}\{\bar{g}(s)\}$ is approximated by

$$g(t) = \frac{\exp(bt)}{T}\left[\frac{1}{2}\bar{g}(b) + \mathrm{Re}\left(\sum_{k=1}^{N} \bar{g}\left(b + \frac{ik\pi}{T}\right)\exp\left(\frac{ik\pi t}{T}\right)\right)\right], \qquad 0 < t < 2T$$

where N is a sufficiently large integer chosen such that

$$\exp(bt)\mathrm{Re}\left[\bar{g}\left(b+\frac{iN\pi}{T}\right)\exp\left(\frac{iN\pi t}{T}\right)\right]\prec\varepsilon$$

The quantity ε is a small positive number that corresponds to the degree of accuracy required. The parameter b is a positive free parameter that must be greater than real parts of all singularities of $\bar{g}(s)$.

4. DISCUSSION OF THE NUMERICAL RESULTS

In this section, we present the obtained numerical results graphically for the variation of the velocity through the mean of the inversion numerical technique outlined above. Two different cases are considered.

Case1: Flow due to the sudden constant motion of the upper plate

In this case, we assume that the upper plate is moving with a constant velocity $v_0 H(t)$, where $H(t)$ is the Heaviside unit step function, defined by

$$H(t)=\begin{cases}1 & if\ t\geq 0\\ 0 & if\ t<0\end{cases}$$

Fig.1 shows the distribution of the velocity versus the distance y between the two plates for different times when the no-slip at the boundaries is assumed and the couple stress coefficient is kept fixed. It can be observed that, for any fixed y, the velocity increases as the time increases and the steady state case is obtained when the parameter t is very large. Fig.2 represents the behavior of the velocity with respect to the distance y as the dimensionless slip velocity parameter of the upper plate α_1 increases, for a fixed t=0.5, couple stress coefficient η=0.01and very large slip parameter α_2 of the stationary plate (No-slip at the lower plate).It can be seen that the increase of α_1 increases the value of the velocity. Also, the classical case of no-slip can be obtained when the slip parameter α_1 tends to infinity. From fig.3, the variation of the velocity for diverse values of slip velocity parameter α_2 of the lower plate is plotted when t=0.5, η=0.01 and large α_1. It can be observed that as the slip velocity parameter of the stationary plate increases there is a decrease in the velocity. Fig.4 depicts the velocity profile for different values of the couple stress parameter when the other parameters are fixed. The graph shows that as the couple stress parameter increases the velocity increases too.

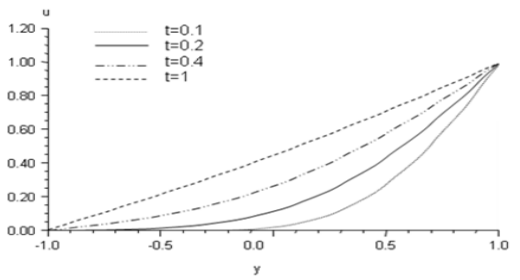

Figure-1. Velocity variation versus distance at η= 0.01.α_1 α_2 =∞ and at different times for case 1

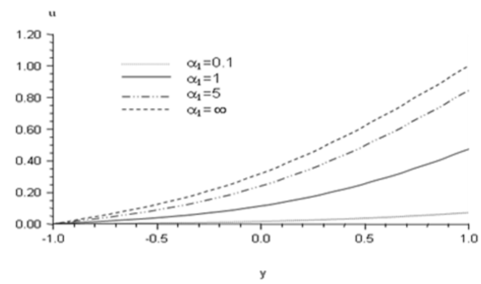

Figure-2. Velocity variation versus distance at η= 0.01.α_1 α_2 =∞ t=5 and at different α_1 for case 1

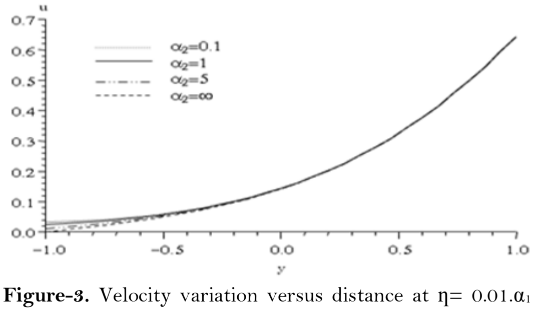

Figure-3. Velocity variation versus distance at η= 0.01.α_1 α_2 =∞ t=5 and at different α_2 for case 1

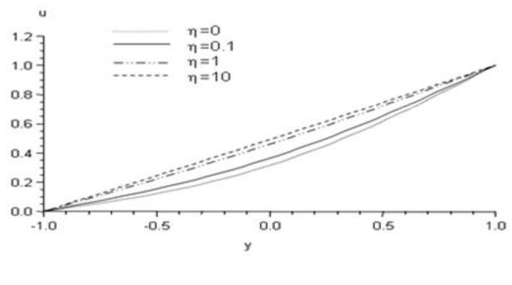

Figure-4. Velocity variation versus distance at η= 0.01.α_1 α_2 =∞ t=5 and at different η for case 1

Case2: Flow due to the sudden oscillatory motion of the upper plate

In this case, we assume that the upper plate is oscillating with a velocity $v_0 \sin t$. In the figures 5-8, we study the variation of the velocity for different times and for different slip and couple stress viscosity parameters. The effects of these parameters are similar to the earlier one.

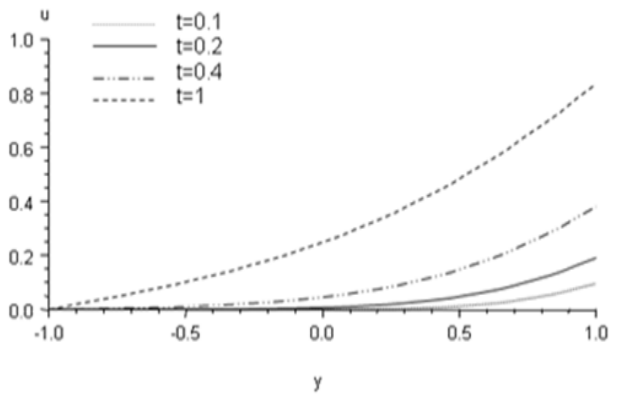

Figure-5. Velocity variation versus distance at η= 0.01.α_1 α_2 =∞. and at different times for case 2

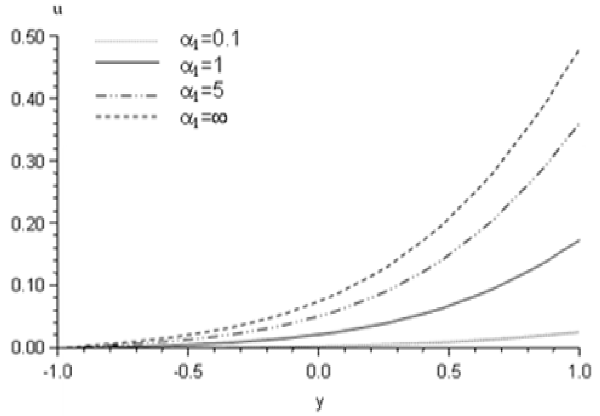

Figure-6. Velocity variation versus distance at η= 0.01.α_1 α_2 =∞. t=0.5 and at different α_1 for case 2

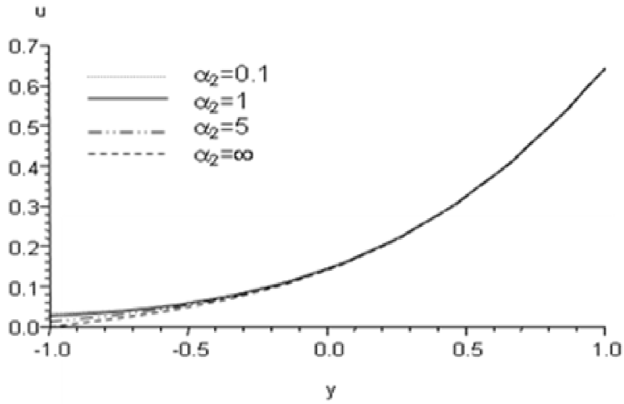

Figure-7. Velocity variation versus distance at η= 0.01.α_1 =∞. t=0.5 and at different α_2 for case 2

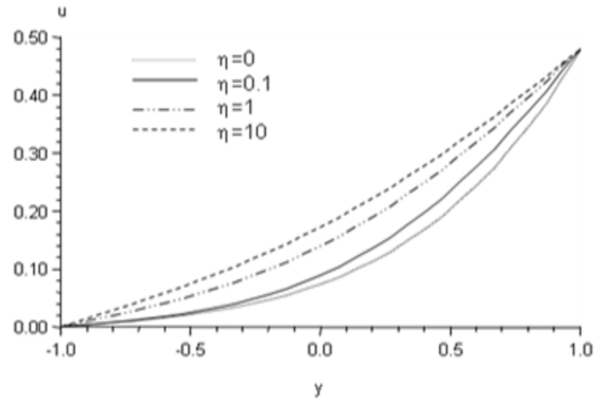

Figure-8. Velocity variation versus distance at η= 0.01.α_1 α_2 =∞. 0.5 and at different η for case 2

5. CONCLUSION

The unsteady motion of an incompressible couple stress fluid confined between two parallel horizontal plates of infinite dimensions is investigated, in the absence of the pressure gradient and under the application of the slip condition on the surfaces of both plates. The fluid's flow is due to the sudden motion of the upper plate. The study is carried out for two cases; the first case allows the upper plate to move with a constant velocity in the positive direction of the x-axis, while the second one assumes that the upper plate oscillates tangentially along the x-axis. The analytical expression of the fluid velocity is found in the Laplace domain using the condition that the couple stresses are absent on the boundaries. The velocity field is obtained in the real domain using a standard numerical inversion procedure.

The study showed that the increase of time enhances the magnitude of the fluid velocity and the steady state solution is obtained from the first case when the time takes large values. In addition, it is of interest to see that, near the upper plate, the raise in the slip parameter amplifies the velocity considerably, whereas, at the stationary plate, this trend is reversed, and the decrease is not significant. Besides, the classical case of no-slip is recovered as a special case when the slip parameters tend to infinity. Furthermore, the presence of couple stress parameter causes notable increase in the velocity field, and the case of classical viscous fluid can be obtained from this work when the couple stress coefficient is taken zero.

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Contributors/Acknowledgement: Both authors contributed equally to the conception and design of the study.

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