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RLS FIXED-LAG SMOOTHER USING COVARIANCE INFORMATION BASED ON INNOVATION APPROACH IN LINEAR CONTINUOUS STOCHASTIC SYSTEMS

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ABSTRACT

This paper newly designs the RLS (recursive least-squares) fixed-lag smoother and filter, based on the innovation theory, in linear continuous-time stochastic systems. It is assumed that the signal is observed with additive white noise and the signal is uncorrelated with the observation noise. It is a characteristic that the estimators use the covariance information of the signal, in the form of the semi-degenerate kernel, and the observation noise. With respect to the RLS fixed-lag smoother, the algorithm for the estimation error variance function is developed to guarantee the stability of the fixed-lag smoother. The proposed estimators have the recursive property in calculating the fixed-lag smoothing and filtering estimates. Also, this paper proposes the Chandrasekhar-type RLS Wiener filter in linear wide-sense stationary stochastic system. Unlike the usual filter including the Riccati-type equations, the Chandrasekhar-type filter does not contain the Riccati-type differential equations and has an advantage of eliminating the possibility of the covariance matrix becoming nonnegative.

Keywords: Linear continuous systems, Fixed-lag smoother, RLS estimation problem, Covariance information, Wiener-Hopf integral equation, Stochastic signal.

1. INTRODUCTION

There are three types of smoothers, which are the fixed-interval smoother, fixed-point smoother and the fixed-lag smoother. It is well-known that the Kalman estimators use the state-space model of the signal to be estimated. Alternatively, instead of the state-space model for the signal, there are estimators, which use the covariance information of the signal and the observation noise. Of the three types of smoothers, this paper focuses on the fixed-lag smoother. Hitherto, the fixed-lag smoothing algorithms have been studied mainly in linear discrete-time stochastic systems, e.g. the RLS (recursive-least squares) fixed-lag smoother using the covariance information [1] the fixed-lag smoother [2] based on the innovation theory, using the covariance information, and the RLS Wiener fixed-lag smoother [3]. Also, in linear continuous-time systems, the recursive fixed-lag smoother, using the covariance information, is designed assuming that the smoothing estimate is given as a linear integral transformation of the observation process [4]; [5].

In linear estimation theory, we often encounter the approach based on the innovation theory. From the viewpoint of the innovation approach, this paper, as a new attempt, examines to design the RLS fixed-lag smoother and the filter using the covariance information. The fixed-lag smoother and the filter calculate the estimates recursively. In this paper, it is assumed that the signal is observed with additive white noise and the processes of the signal and the observation noise are mutually independent. The estimation problem for the signal process correlated with the colored observation noise can be dealt with as an extension of the independent case between the processes of the signal and the white observation noise.

This paper, based on the RLS filter presented in this paper, further develops the Chandrasekhar-type filter from the RLS Wiener filter. The RLS Wiener filter [6] is derived from the RLS filter using the covariance information. The derivation of the Chandrasekhar-type filter is similar to the technique in Kailath [7] for the Chandrasekhar-type filter from the Kalman filter with the state-space model. It is known that the Chandrasekhar-type filter is advantageous to the RLS Wiener filter from the aspect of not including the Riccati-type differential equations. The Riccati-type differential equations generate the round-off errors, which sometimes result in the instability of the filtering estimate particularly for the small value of the observation noise variance.

Two numerical simulation examples are demonstrated to show the estimation property of the proposed RLS fixed-lag smoother using the covariance information.

2. FIXED-LAG SMOOTHING PROBLEM

Let an observation equation be given by

$$y(t) = z(t) + v(t) \qquad (1)$$

in linear continuous-time stochastic systems, where $z(t)$ is an $\mathrm{m} \times 1$ signal vector and $v(t)$ is white observation noise. It is assumed that the signal and the observation noise are mutually independent stochastic processes with zero means. Let the auto-covariance function of $v(t)$ be given by

$$E[v(t)v^T(s)] = R\delta(t-s), \quad R > 0. \qquad (2)$$

Here, $\delta(\cdot)$ denotes the Dirac δ function.

Let $K(t,s)$ represent the auto-covariance function of the signal and let $K(t,s)$ be expressed in the semi-degenerate kernel [4] form of

$$K(t,s) = \begin{cases} A(t)B^T(s), & 0 \le s \le t, \\ B(t)A^T(s), & 0 \le t \le s. \end{cases} \qquad (3)$$

Here, $A(t)$ and $B(s)$ are bounded $\mathrm{m} \times p$ matrices.

Let a fixed-lag smoothing estimate $\hat{z}(t, t+D)$ of $\mathrm{z}(t)$ be given by

$$\hat{z}(t, t+D) = \int_0^{t+D} g\,(t, s)\upsilon(s)ds \tag{4}$$

as a linear integral transformation of the innovation process $\upsilon(s) = y(s) - \hat{z}(s, s)$, $0 \le s \le t + D$, where $g(t, s)$, D and $\hat{z}(s, s)$ are referred to be an impulse response function, the fixed lag and the filtering estimate of $z(s)$.

The impulse response function, which minimizes the mean-square value of the fixed-lag smoothing error $z(t) - \hat{z}(t, t+D)$,

$$J = E[\|z(t) - \hat{z}(t, t+D)\|^2], \tag{5}$$

satisfies

$$\begin{aligned} E[z(t)v^T(s)] &= \int_0^{t+D} g\,(t, \tau)E[v(\tau)v^T(s)]d\tau \\ &= g(t, s)R \end{aligned} \tag{6}$$

by an orthogonal projection lemma [1]; [2]:

$$z(t) - \hat{z}(t, t+D) \perp v(s), \quad 0 \le s \le t. \tag{7}$$

Here, "⊥" denotes the notation of the orthogonality. From (1), (2) and (6), the linear least-squares impulse response function satisfies

$$\begin{aligned} g(t, s)R &= E[x(t)(y(s) - \hat{z}(s, s))^T] \\ &= K(t, s) - \int_0^s E[x(t)v^T(\tau)]\, g^T(s, \tau)d\tau \\ &= K(t, s) - \int_0^s g(t, \tau)R\, g^T(s, \tau)d\tau. \end{aligned} \tag{8}$$

3. FIXED-LAG SMOOTHING AND FILTERING ALGORITHMS

The expression for the fixed-lag smoothing estimate in (4) might be written as

$$\hat{z}(t, t+D) = \int_0^t g\,(t, s)v(s)ds + \int_t^{t+D} g\,(t, s)v(s)ds. \tag{9}$$

The first term on the right hand side represents the filtering estimate $\hat{z}(t, t)$ of $z(t)$ and the second term represents the correction term of the fixed-lag smoothing estimate to the filtering estimate. For $0 \le s \le t$, from (3), the impulse response function $g(t, s)$ satisfies

$$g(t, s)R = A(t)B^T(s) - \int_0^s g\,(t, \tau)R(\tau)g^T(s, \tau)d\tau. \tag{10}$$

Introducing an auxiliary function $J(s)$, which satisfies

$$J(s)R = B^T(s) - \int_0^s J\,(\tau)R(\tau)g^T(s, \tau)d\tau, \tag{11}$$

we obtain

$$g(t, s) = A(t)J(s). \tag{12}$$

Substituting (12) into (11) and introducing a function

$$r(s) = \int_0^s J(\tau)R\,J^T(\tau)d\tau, \tag{13}$$

we have

$$J(s)R = B^T(s) - \int_0^s J(\tau)R(\tau)J^T(\tau)A^T(s)d\tau$$
$$= B^T(s) - r(s)A^T(s). \tag{14}$$

Differentiating (13) with respect to s, we obtain

$$\frac{dr(s)}{ds} = J(s)RJ^T(s), r(0) = 0. \tag{15}$$

Let us introduce a function

$$\mathrm{e(t)} = \int_0^t J(\tau)\nu(\tau)d\tau. \tag{16}$$

From (9) and (12), the filtering estimate $\hat{z}(t,t)$ is given by

$$\hat{z}(t,t) = \int_0^t g(t,\tau)\nu(\tau)d\tau$$
$$= \int_0^t A(t)J(\tau)\,\nu(\tau)d\tau \tag{17}$$
$$= A(t)e(t).$$

Differentiating (16) with respect to t, we have

$$\frac{de(t)}{dt} = \mathrm{J(t)}\nu(\mathrm{t}) = \mathrm{J(t)}\big(\mathrm{y(t)} - \hat{z}(t,t)\big), \mathrm{e(0)} = 0. \tag{18}$$

For $\mathrm{t} \leq \mathrm{s} \leq \mathrm{t} + \mathrm{D}$, from (3), (8) is written as

$$g(t,s)R(s) = B(t)A^T(s) - \int_0^s g(t,\tau)R\,g^T(s,\tau)d\tau. \tag{19}$$

Introducing an auxiliary function $\tilde{J}(s)$, which satisfies

$$\tilde{J}(s)R = A^T(s) - \int_0^s \tilde{J}(\tau)R\,g^T(s,\tau)d\tau, \tag{20}$$

we obtain

$$g(t,s) = B(t)\tilde{J}(s), 0 \leq s \leq t. \tag{21}$$

Introducing a function

$$\tilde{r}(s) = \int_0^s \tilde{J}(\tau)RJ^T(\tau)d\tau, \tag{22}$$

from (12) and (20), $\tilde{J}(s)$ is expressed by

$$\tilde{J}(s)R = A^T(s) - \int_0^s \tilde{J}(\tau)R(\tau)\,g^T(s,\tau)d\tau$$
$$= A^T(s) - \int_0^s \tilde{J}(\tau)R(\tau)J^T(\tau)A^T(s)d\tau \tag{23}$$
$$= A^T(s) - \tilde{r}(s)A^T(s).$$

Differentiating (22) with respect to s, we have

$$\frac{d\tilde{r}(s)}{ds} = \tilde{J}(s)RJ^T(s), \tilde{r}(0) = 0. \tag{24}$$

Let us introduce a function

$$\tilde{e}(t,t+D) = \int_t^{t+D} \tilde{J}(s)\nu(s)ds. \tag{25}$$

From (9), the fixed-lag smoothing estimate $\hat{z}(t,t+D)$ of $\mathrm{z(t)}$ is expressed as

$$\hat{z}(t,t+D) = \hat{z}(t,t) + \int_t^{t+D} B(t)\tilde{J}(s)v(s)ds$$
$$= \hat{z}(t,t) + B(t)\tilde{e}(t,t+D). \tag{26}$$

Differentiating (25) with respect to t, we have

$$\frac{d\tilde{e}(t,t+D)}{dt} = \tilde{J}(t+D)(y(t+D) - \hat{z}(t+D,t+D)) - \tilde{J}(t)(y(t) - \hat{z}(t,t)). \tag{27}$$

The initial condition on the differential equation (27) for $\tilde{e}(t,t+D)$ at $t = 0$ is $\tilde{e}(0,D)$, which is specified by

$$\tilde{e}(0,D) = \int_0^D \tilde{J}(s)v(s)ds. \tag{28}$$

Differentiating (28) with respect to D, we obtain

$$\frac{d\tilde{e}(0,D)}{dD} = \tilde{J}(D)\big(y(D) - \hat{z}(D,D)\big), \ \ \tilde{e}(0,0) = 0. \tag{29}$$

Now, let us summarize the fixed-lag smoothing algorithm in Theorem 1.

Theorem 1 Let the observation equation be given by (1). Let the auto-covariance function of the signal $z(t)$ be given by (3) in the semi-degenerate kernel form in linear continuous-time stochastic systems. Then the fixed-lag smoothing estimate $\hat{z}(t,t+D)$ of $z(t)$ is calculated recursively by (30)-(37). Fixed-lag smoothing estimate $\hat{z}(t,t+D)$ of z(t):

$$\hat{z}(t,t+D) = \hat{z}(t,t) + B(t)\tilde{e}(t,t+D) \tag{30}$$

$$\frac{d\tilde{e}(t,t+D)}{dt} = \tilde{J}(t+D)(y(t+D) - \hat{z}(t+D,t+D)) - \tilde{J}(t)(y(t) - \hat{z}(t,t)) \tag{31}$$

$$\tilde{J}(t) = (A^T(t) - \tilde{r}(t)A^T(t))R^{-1} \tag{32}$$

$$\frac{d\tilde{r}(t)}{dt} = \tilde{J}(t)RJ^T(t), \tilde{r}(0) = 0 \tag{33}$$

Filtering estimate $\hat{z}(t,t)$ of z(t):

$$\hat{z}(t,t) = A(t)e(t) \tag{34}$$

$$\frac{de(t)}{dt} = \mathrm{J(t)}\big(\mathrm{y(t)} - \hat{z}(t,t)\big), \mathrm{e}(0) = 0 \tag{35}$$

$$\frac{dr(t)}{dt} = J(t)RJ^T(t), r(0) = 0 \tag{36}$$

$$J(t) = (B^T(t) - r(t)A^T(t))R^{-1} \tag{37}$$

Initial condition on the differential equation (31) for $\tilde{e}(t,t+D)$ at $t = 0$ is $\tilde{e}(0,D)$, which satisfies the differential equation

$$\frac{d\tilde{e}(0,D)}{dD} = \tilde{J}(D)\big(y(D) - \hat{z}(D,D)\big), \ \ \tilde{e}(0,0) = 0. \tag{38}$$

4. CHANDRASEKHAR-TYPE RLS WIENER FILTER

It is known that the Chandrasekhar-type filter is advantageous to the Kalman filter from the numerical aspect [7]. Since the Chandrasekhar-type filter does not include the Riccati-type differential equations, the round-off errors are avoided in the computation of the Riccati-type differential equations.

This section, in the first place, derives the RLS Wiener filter from the RLS filter using the covariance information. In the second place, the Chandrasekhar-type RLS Wiener filter is obtained from the RLS Wiener filter.

Concerning (1), we introduce the observation matrix H as $\mathrm{z(t) = Hx(t)}$. Here, $\mathrm{x(t)}$ denotes the $\mathrm{n \times 1}$ state vector. By introducing the system matrix F for $\mathrm{x(t)}$, from (34) and using (35), it is clear that the filtering estimate $\hat{x}(t,t)$ of the state $\mathrm{x(t)}$ is calculated by

$$\frac{d\hat{x}(t,t)}{dt} = F\hat{x}(t,t) + e^{Ft}J(t)\big(y(t) - \hat{z}(t,t)\big). \tag{39}$$

Let $K_x(t,t)$ denote the variance of the state $\mathrm{x(t)}$ and $\mathrm{G(t) = A(t)J(t)} = e^{Ft}J(t)$ denote the filter gain. By introducing the function $\mathrm{S(t)} = e^{Ft}r(t)(e^{Ft})^T$, the filter gain $\mathrm{G(t)}$ is expressed as

$$G(t) = (K_x(t,t)H^T - S(t)H^T)R^{-1}. \tag{40}$$

Differentiating $\mathrm{S(t)}$ with respect to t, from (36), we have

$$\begin{aligned} \frac{dS(t)}{dt} &= \frac{dA(t)}{dt}r(t)A^T(t) + A(t)r(t)\frac{dA^T(t)}{dt} \\ &\quad + A(t)J(t)RJ^T(t)A^T(t) \\ &= FS(t) + S(t)F^T + G(t)RG^T(t),\ \mathrm{S(0)=0}. \end{aligned} \tag{41}$$

Differentiating (41) with respect to t, we have

$$\frac{d^2S(t)}{dt} = F\frac{dS(t)}{dt} + \frac{dS(t)}{dt}F^T + \frac{dG(t)}{dt}RG^T(t) + G(t)R\frac{dG^T(t)}{dt}. \tag{42}$$

By the way, Differentiating (40) with respect to t, we have

$$\frac{dG(t)}{dt} = -\frac{dS(t)}{dt}H^TR^{-1}. \tag{43}$$

Here, $\frac{dK_x(t,t)}{dt} = 0$ is clear from the property $K_x(t,t) = K_x(t-t) = K_x(0)$ for the autovariance function of the state $\mathrm{x(t)}$ in linear wide-sense stationary stochastic systems. Substituting (43) into (42), we obtain

$$\frac{d^2S(t)}{dt} = (F - G(t)H)\frac{dS(t)}{dt} + \frac{dS(t)}{dt}(F - G(t)H)^T. \tag{44}$$

Let $\Phi(\mathrm{t},0)$ be the state-transition matrix for $\mathrm{x(t)}$. It is found that $\frac{dS(t)}{dt}$ is given by

$$\begin{aligned} \frac{dS(t)}{dt} &= \Phi(t,0)\frac{dS(t)}{dt}\Big|_{t=0}\Phi^T(t,0), \\ \frac{d\Phi(t,0)}{dt} &= (\mathrm{F} - \mathrm{G(t)}H)\Phi(t,0), \Phi(0,0) = I. \end{aligned} \tag{45}$$

From (41), $\frac{dS(t)}{dt}\big|_{t=0}$ is given by $\frac{dS(t)}{dt}\big|_{t=0} = G(0)RG^T(0)$. Hence, from (45), it is clear that

$$\frac{dS(t)}{dt} = \Phi(t,0)G(0)RG^T(0)\Phi^T(t,0)$$
$$= \Phi(t,0)G(0)R^{\frac{1}{2}}R^{\frac{1}{2}}G^T(0)\Phi^T(t,0) \quad (46)$$
$$= L(t)L^T(t), L(t) = \Phi(t,0)G(0)R^{\frac{1}{2}}.$$

The differential equation for $L(t)$ is given by

$$\frac{dL(t)}{dt} = (F - G(t)H)L(t), L(0) = G(0)R^{\frac{1}{2}}. \quad (47)$$

From (40), the initial condition on the differential equation (43) at $t = 0$ is specified by

$$G(0) = K_x(0)H^T R^{-1}. \quad (48)$$

Substituting $\frac{dS(t)}{dt} = L(t)L^T(t)$ into (43), we obtain

$$\frac{dG(t)}{dt} = -L(t)L^T(t)H^T R^{-1}. \quad (49)$$

As a result, the Chandrasekhar-type RLS Wiener filtering algorithm for the filtering estimate $\hat{z}(t,t)$ of the signal $z(t)$ is summarized in Theorem 2. Theorem 2 Let the observation equation be given by (1). Let the signal $z(t)$ is related with the state as $z(t) = Hx(t)$. Then the RLS Wiener filtering algorithm consists of (50)-(53) in linear continuous-time stochastic systems.

Filtering estimate $\hat{z}(t,t)$ of the signal $z(t)$: $\hat{z}(t,t) = H\hat{x}(t,t)$

$$\hat{z}(t,t) = H\hat{x}(t,t) \quad (50)$$

Filtering estimate $\hat{x}(t,t)$ of the state $x(t)$:

$$\frac{d\hat{x}(t,t)}{dt} = F\hat{x}(t,t) + G(t)\big(y(t) - H\hat{x}(t,t)\big), \hat{x}(0,0) = 0 \quad (51)$$

Filter gain: $G(t)$

$$G(t) = (K_x(t,t)H^T - S(t)H^T)R^{-1}. \quad (52)$$

Riccati-type differential equation for the variance $S(t)$ of the filtering estimate $\hat{x}(t,t)$:

$$\frac{dS(t)}{dt} = FS(t) + S(t)F^T + G(t)RG^T(t), S(0) = 0 \quad (53)$$

Also, the Chandrasekhar-type RLS Wiener filtering algorithm for the filtering estimate $\hat{z}(t,t)$ of the signal $z(t)$ consists of (50), (51), (54) and (55) in linear continuous-time wide-sense stationary stochastic systems.

Filter gain: $G(t)$

$$\frac{dG(t)}{dt} = -L(t)L^T(t)H^T R^{-1}, G(0) = K_x(0)H^T R^{-1} \quad (54)$$

Differential equation for $L(t)$:

$$\frac{dL(t)}{dt} = (F - G(t)H)L(t), L(0) = K_x(0)H^T R^{-\frac{1}{2}} \quad (55)$$

It is a characteristic that (54) calculates the filter gain of the Chandrasekhar-type RLS Wiener filter. Whereas (52) calculates the filter gain of the RLS Wiener filter by use of $S(t)$, which is computed by the Riccati-type differential equations of (53). Since $G(t)$ and $L(t)$ are $n \times m$ matrices, the number of the differential equations to get the filter gain $G(t)$ by (54) and (55) is $2nm$ in the Chandrasekhar-type RLS Wiener filter. The number of the Riccati-type differential equations is $n(n+1)/2$ in the RLS Wiener filter. Compared with the RLS Wiener filter, in addition to the less number of differential equations, under the inequality condition

$\mathrm{m} < \frac{(\mathrm{n}+1)}{4}$, the Chandrasekhar-type RLS Wiener filter has an advantage of eliminating the possibility of the covariance matrix becoming nonnegative due to round-off errors in the calculation of the filtering estimate.

Based on the state-space model in linear continuous-time varying stochastic systems, the Chandrasekhar-type filter is described in Baras and Lainiotis [8]. It should be noted that the Chandrasekhar-type filter in Theorem 2 is different from those in Kailath [7]; Baras and Lainiotis [8] (see page 165) from the viewpoint that the current Chandrasekhar-type filter is derived from the RLS Wiener filter using the system matrix F, the variance $K_x(0)$ of the state $\mathrm{x(t)}$, the observation matrix H and the variance R of the observation noise.

5. FIXED-LAG SMOOTHING ERROR VARIANCE FUNCTION

Let $P_{\tilde{z}}(t-D,t)$ denote the fixed-lag smoothing error variance function as

$$\begin{aligned} P_{\tilde{z}}(t,t+D) &= E[(z(t)-\hat{z}(t,t+D))(z(t)-\hat{z}(t,t+D))^T] \\ &= K(t,t) - E[\hat{z}(t,t+D)\hat{z}^T(t,t+D)]. \end{aligned} \tag{56}$$

From (30) and (34), we have

$$\begin{aligned} &E[\hat{z}(t,t+D)\hat{z}^T(t,t+D)] \\ &\qquad = E[(A(t)e(t)+B(t)\tilde{e}(t,t+D))(A(t)e(t) \\ &\qquad + B(t)\tilde{e}(t,t+D))^T]. \end{aligned} \tag{57}$$

By introducing the following functions

$$\begin{aligned} f_1(t) &= E[e(t)e^T(t)], \\ f_2(t,t+D) &= E[\tilde{e}(t,t+D)(t)\tilde{e}^T(t,t+D)], \end{aligned} \tag{58}$$

(56) is written as

$$P_{\tilde{z}}(t,t+D) = K(t,t) - A(t)f_1(t)A^T(t) - B(t)f_2(t,t+D)B^T(t). \tag{59}$$

From (13) and (16), $f_1(t)$ is equivalent to $r(t)$ as shown by

$$\begin{aligned} f_1(t) &= \mathrm{E}[\int_0^t J(\tau)v(\tau)d\tau(\int_0^t J(s)v(s))^T] \\ &= \int_0^t J(s)RJ^T(s)ds \\ &= r(t). \end{aligned} \tag{60}$$

Also, from (58) with (25), $f_2(t)$ is expressed by

$$\begin{aligned} f_2(t,t+D) &= E[\int_t^{t+D} \tilde{J}(\tau)v(\tau)d\tau(\int_t^{t+D} \tilde{J}(s)v(s)ds)^T] \\ &= \int_t^{t+D} \tilde{J}(s)\, R\tilde{J}^T(s)ds. \end{aligned} \tag{61}$$

Differentiating (61) with respect to t, we obtain

$$\frac{df_2(t,t+D)}{dt} = \tilde{J}(t+D)R\tilde{J}^T(t+D) - \tilde{J}(t)R\tilde{J}^T(t). \tag{62}$$

The initial condition on the differential equation (62) for $f_2(t,t+D)$ at $\mathrm{t}=0$ is $f_2(0,D)$. From (61), $f_2(0,D)$ satisfies the differential equation.

$$\frac{df_2(0,D)}{dD} = \tilde{J}(D)R\tilde{J}^T(D), f_2(0,0) = 0. \tag{63}$$

Hence, the fixed-lag smoothing error variance function $P_{\tilde{z}}(t, t+D)$ is calculated by (59) together with (32), (33), (36), (37), (60), (62) and (63) recursively.

From (9), (56) and (58), it is seen that the fixed-lag smoothing error variance is lower bounded by the zero matrix and upper bonded by the filtering error variance function as

$$0 \leq P_{\tilde{z}}(t, t+D) \leq= K(t,t) - A(t) f_1(t) A^T(t). \tag{64}$$

(64) indicates that the estimation accuracy of the proposed RLS fixed-lag smoother is equal to or better than that of the RLS Wiener filter in Theorem 1.

The numerical aspects of the proposed fixed-lag smoother and filter are examined in section 6.

6. NUMERICAL SIMULATION EXAMPLES

6.1. Example 1

Let a scalar observation equation be given by

$$y(t) = z(t) + v(t). \tag{65}$$

Let the observation noise $v(t)$ be a zero-mean white Gaussian process with the variance R, $N(0, R)$. Let the auto-covariance function of the signal $z(t)$ be given by

$$K(t,s) = \frac{3}{16} e^{-|t-s|} + \frac{5}{48} e^{-3|t-s|}. \tag{66}$$

From (66), the functions $A(t)$ and $B(s)$ in (3) are expressed as follows:

$$A(t) = [\frac{3}{16} e^{-t} \quad \frac{5}{48} e^{-3t}], \quad B(s) = [e^{s} \quad e^{3s}]. \tag{67}$$

If we substitute (67) into the fixed-lag smoothing algorithm of Theorem 1, we can calculate the fixed-lag smoothing estimate recursively. Fig.1 illustrates the signal $z(t)$ and the fixed-lag smoothing estimate $\hat{z}(t, t+0.01, t)$ for the white Gaussian observation noise $N(0, 0.1^2)$ by the RLS fixed-lag smoother in Theorem1. It is shown that the fixed-lag smoothing estimate $\hat{z}(t, t+0.01, t)$ approaches the signal $z(t)$ gradually as time increases. From almost time $\mathrm{t} = 1.4$ on, the fixed-lag smoothing estimate converges almost to the signal with a few estimation errors. Fig.2 illustrates the MSVs (mean-square values) of the fixed-lag smoothing and filtering errors by the proposed RLS estimators in Theorem 1 for the observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$ and $N(0, 0.5^2)$ vs. the fixed lag D, $0 \leq D \leq 10\Delta$. For $D = 0$, the MSV of the filtering error is shown. From Fig.2, we note that the estimation accuracies of the fixed-lag smoother and filter in Theorem 1 are almost equal. Also, the smaller the observation noise variance becomes, the better the estimation accuracies of the fixed-lag smoother and the filter become.

Here, the MSVs of the fixed-lag smoothing and filtering errors are evaluated by $\sum_{i=1}^{2000}(z(\Delta i) - z(\Delta i, \Delta i + D))^2/2000$ and $\sum_{i=1}^{2000}(z(\Delta i) - z(\Delta i, \Delta i))^2/2000$, $\Delta = 0.001$. Also, as the numerical integration of the differential equations, the fourth-order Runge-Kutta method is used. For references, the state-space model, which generates the signal process, is specified by

$$z(t) = x_1(t),$$

$$\frac{dx_1(t)}{dt} = x_2(t) + u(t), \quad \frac{dx_2(t)}{dt} = -3x_1(t) - 4x_2(t) - 2w(t),$$

$$E[w(t)w(s)] = \delta(t-s).$$

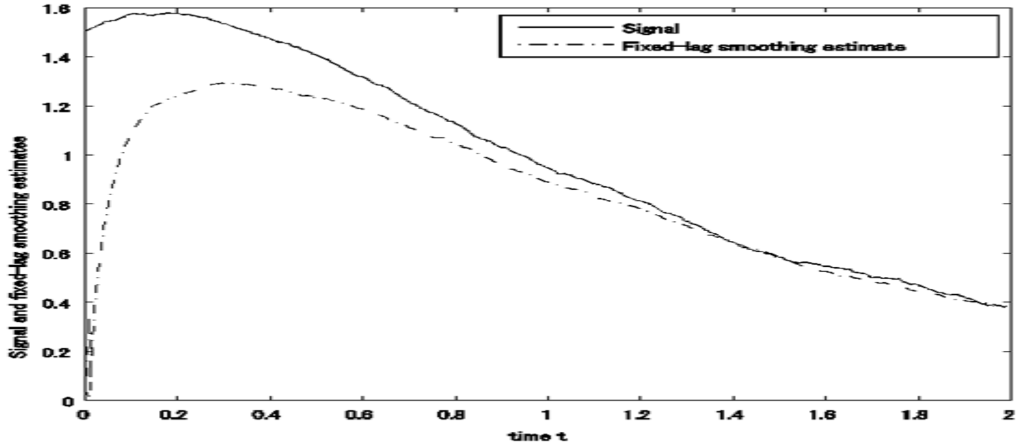

Fig-1. Signal $z(t)$ and the fixed-lag smoothing estimate $\hat{z}(t, t+0.01)$ for the white Gaussian observation noise $N(0,0.1^2)$ by the RLS fixed-lag smoother in Theorem1.

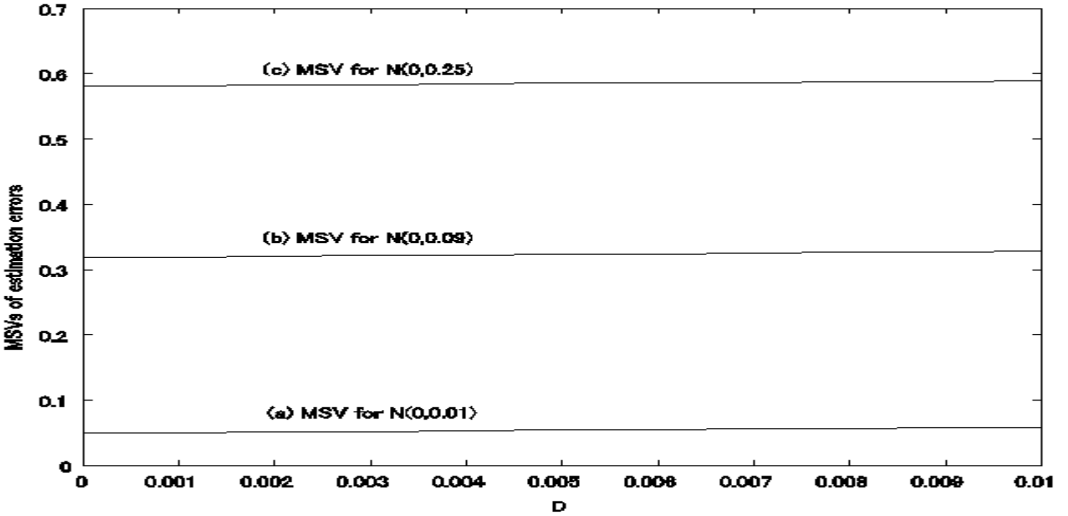

Fig-2. Mean-square values of the fixed-lag smoothing and filtering errors by the proposed RLS estimators in Theorem 1 for the observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$ vs. the fixed lag D, $0 \leq D \leq 10\Delta, \Delta = 0.001$.

6.2. Example 2

Under the same assumptions on the observation equation (64) and the observation noise as section 6.1, section 6.1 provides the second simulation example.

Let the auto-covariance function of the signal $z(t)$ be given by

$$K(t,s) = \frac{5}{3}e^{-|t-s|} - \frac{5}{6}e^{-2|t-s|}. \quad (68)$$

From (68), the functions $A(t)$ and $B(s)$ in (3) are expressed as follows:

$$A(t) = [\frac{5}{3}e^{-t} \quad -\frac{5}{6}e^{-2t}], \quad B(s) = [e^{s} \quad e^{2s}]. \quad (69)$$

If we substitute (69) into the fixed-lag smoothing algorithm of Theorem 1, we can calculate the fixed-lag smoothing estimate recursively. Fig.3 illustrates the signal $z(t)$ and the fixed-lag smoothing estimate $\hat{z}(t, t+0.01)$ for the white Gaussian observation noise $N(0,0.1^2)$ by the RLS fixed-lag smoother in Theorem1. As in section 6.1, the fixed-lag smoothing estimate converges to the signal as time advances almost after $\mathrm{t} = 0.6$. Fig.4 illustrates the MSVs of the fixed-lag smoothing and filtering errors by the proposed RLS estimators in Theorem 1 for the observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$ vs. the fixed lag D, $0 \le D \le 10\Delta$. Fig.4 indicates that the estimation accuracies of the fixed-lag smoother and filter are almost equivalent. Also, the smaller the observation noise variance becomes, the better the estimation accuracies of the smoother and the filter become. Table 1 shows the square values of the filtering error $z(1) - \hat{z}(1,1)$ and of the fixed-lag smoothing errors $z(1) - \hat{z}(1,1 + 0.005)$ and $z(1) - \hat{z}(1,1 + 0.01)$ for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$. The mean value of the fixed-lag smoothing error $z(1) - \hat{z}(1,1 + 0.005)$ is less than the filtering error $z(1) - \hat{z}(1,1)$ respectively for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$. This indicates that the, compared with the filtering estimate, the fixed-lag smoothing estimate approaches the signal z(1)=1.155824166469756 by using the observed values $\mathrm{y}(1 + 0.001 \times \mathrm{i})$, $1 \le \mathrm{i} \le 5$.

The MSVs of the fixed-lag smoothing and filtering errors are evaluated same as in section 6.1.

For references, the state-space model, which generates the signal process, is specified by

$$z(t) = x_1(t),$$
$$\frac{dx_1(t)}{dt} = x_2(t), \quad \frac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) + w(t),$$
$$E[w(t)w(s)] = 5\delta(t - s).$$

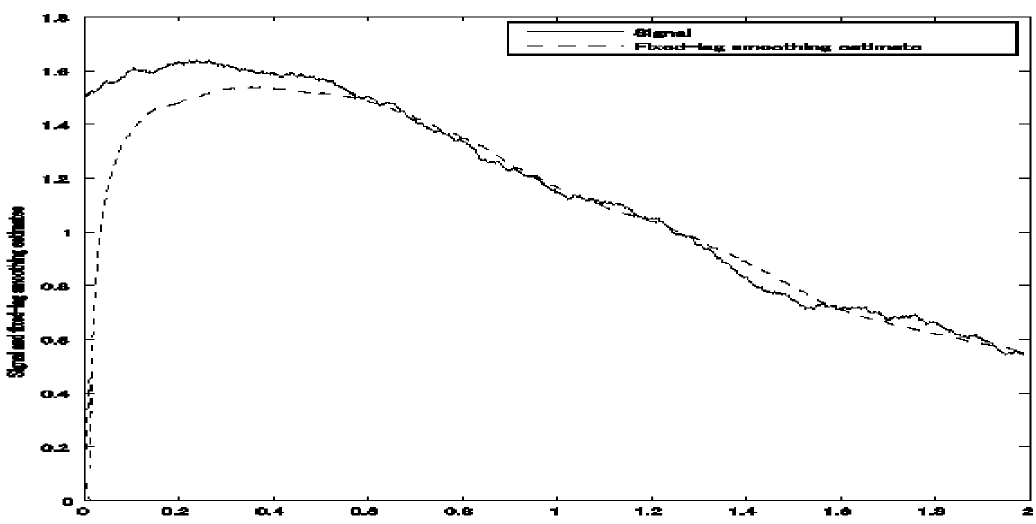

Fig-3. Signal $z(t)$ and the fixed-lag smoothing estimate $\hat{z}(t, t + 0.01)$ for the white Gaussian observation noise $N(0,0.1^2)$ by the RLS fixed-lag smoother in Theorem1.

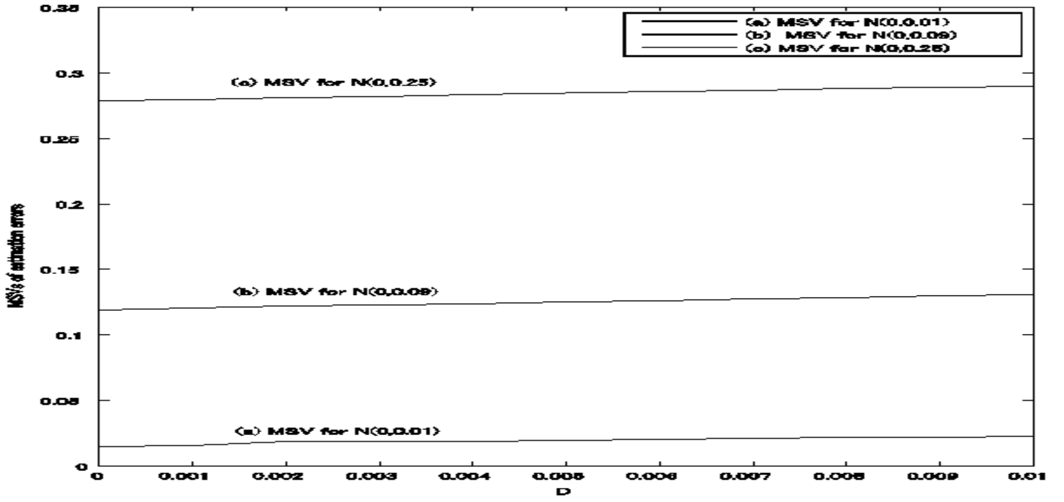

Fig-4. Mean-square values of the fixed-lag smoothing and filtering errors by the proposed RLS estimators in Theorem 1 for the observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$ vs. the fixed lag D, $0 \leq D \leq 10\Delta, \Delta = 0.001$.

Table-1. Square values of the filtering error $z(1) - \hat{z}(1,1)$ and of the fixed-lag smoothing errors $z(1) - \hat{z}(1,1 + 0.005)$ and $z(1) - \hat{z}(1,1 + 0.01)$ for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$ and $N(0,0.5^2)$.

	$(z(1) - \hat{z}(1,1))^2$	$(z(1) - \hat{z}(1,1 + 0.005))^2$	$(z(1) - \hat{z}(1,1 + 0.01))^2$
$N(0,0.1^2)$	1.143359201822944e-04	1.081306627506604e-04	3.277422949476264e-04
$N(0,0.3^2)$	0.001117192449080	8.353781244847178e-04	8.218746244468232e-04
$N(0,0.5^2)$	0.034248303862559	0.033543051422689	0.034752454154317

7. CONCLUSIONS

In this paper, based on the innovation approach, the RLS fixed-lag smoother and filter using the covariance information of the signal, in the form of the semi-degenerate kernel, is newly devised. Furthermore, on the basis of the RLS filter proposed in this paper, the Chandrasekhar-type RLS Wiener filter, with the numerical merits, is proposed. In the RLS Wiener filter, the number of the Riccati-type differential equations is $n(n+1)/2$. Compared with the RLS Wiener filter, in addition to the less number of differential equations, under the inequality condition $m < \frac{(n+1)}{4}$, the Chandrasekhar-type RLS Wiener filter has an advantage of eliminating the possibility of the covariance matrix becoming nonnegative due to round-off errors in the calculation of the filtering estimate.

The numerical simulation examples indicate that the estimation accuracy of the RLS fixed-lag smoother, using the covariance information, is almost equal to that of the RLS Wiener filter. Also, it has been shown that the RLS Wiener fixed-lag smoother, based on the innovation approach, has stable estimation property.

In the current estimators, it is assumed that the signal process and the observation noise process are uncorrelated mutually. However, there is a case where the signal process is observed with additive colored noise, where the processes of the signal and the observation noise are correlated [9]. The estimation problem for the signal correlated with the observation noise is left as a future task.

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