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ON THE STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT

Let f (z) be an analytic function in the open unit disk U normalized with $f(0)=0$ and $f'(0)=1$. In this paper, the starlikeness for $f(z)$ is discussed.

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INTRODUCTION

Let H be the class of analytic functions in $U=\{z\in C:|z|<1\}$, and A be the subclass of H consisting of functions of the form

$$f(z)=z+a_2z^2+a_2z^2+\cdots,z\in U. \qquad (1)$$

A function $f(z)\in A$ is said to be starlike of order $\alpha(0\le\alpha<p)$ in U (see Robertson [1]), that is, $f(z)\in S^*(\alpha)$, if and only if

$$Re(\frac{zf'(z)}{f(z)})>\alpha,0\le\alpha<1,z\in U \qquad (2)$$

with $S_1^*(0):=S^*$.

Similarly, a function $f(z)\in A$ is said to be convex of order $\alpha(0\le\alpha<1)$ in U (see Robertson [1]), that is, $f(z)\in K(\alpha)$, if and only if

$$Re(1+\frac{zf''(z)}{f'(z)})>\alpha, 0\le\alpha<1, z\in U \qquad (3)$$

with $K(0)=K$.

By the definitions for the classes $S^*(\alpha)$ and $K(\alpha)$, we know that $f(z)\in K(\alpha)$ if and only if $f(z)\in S^*(\alpha)$. Marx [2] and Strohhacker [3] showed that $f(z)\in K(0)$ implies $f(z)\in S^*(1/2)$.

Several results appeared previously about sufficient conditions of starlikeness (see Nunokawa, et al. [4] and Sokol [5]). In this paper, With the help of two inequality, the starlikeness for $f(z)$ is discussed.

The Main Results

Lemma 2.1. (see Nunokawa, et al. [6])Let $p(z)=1+c_1z+c_2z^2+\cdots$ be analytic in the unit disc U and $\alpha(0<\alpha\le 1/2)$ be a positive real number. Then suppose that there exists a point $z_0\in U$ such that

$$Rep(z)>\alpha \, for \, |z|<|z_0| \qquad (4)$$

and

$$Rep(z_0)=\alpha, p(z_0)\neq\alpha. \qquad (5)$$

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)}\le-\frac{\alpha}{2(1-\alpha)}. \qquad (6)$$

By using Lemma 2.1, we first prove the following Theorem.

Theorem 2.1. Let $f(z)\in A$, and $\alpha(0<\alpha\le 1/2)$ be a positive real number. Suppose

$$\frac{zf'(z)}{f(z)}\neq\alpha \qquad (7)$$

and

$$Re(1+\frac{zf''(z)}{f'(z)})>Re(\frac{zf'(z)}{f(z)})-\frac{\alpha}{2(1-\alpha)}. \qquad (8)$$

Then we have $f(z)\in S^*(\alpha)$.

Proof. Let

$$p(z)=\frac{zf'(z)}{f(z)}, \tag{9}$$

then $p(z)$ is analytic in U and $p(0)=1$. Suppose that there exists a point $z_0 \in U$ which satisfies the conditions (4) and (5) of Lemma 2.1.

Now using (9), it follows that

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{zp'(z)}{p(z)}. \tag{10}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions Lemma 2.1, therefore in view of (6) and (10) gives

$$Re(1+\frac{z_0 f''(z_0)}{f'(z_0)})=Re(\frac{zp'(z_0)}{p(z_0)}+p(z_0)). \qquad \text{.................(11)}$$

This is a contradiction and therefore proof of the Theorem 2.1 is completed.

Lemma 2.2. (see [6])Let $p(z)=1+c_1z+c_2z^2+\cdots$ be analytic in the unit disc U and $\alpha(1/2<\alpha<1)$ be a positive real number. Then suppose that there exists a point $z_0 \in U$ such that

$$Rep(z)>\alpha \, for \, |z|<|z_0| \tag{12}$$

and

$$Rep(z_0)=\alpha, p(z_0)\neq\alpha. \tag{13}$$

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)}\leq-\frac{1-\alpha}{2\alpha}. \tag{14}$$

By using Lemma 2.2, we can prove the following Theorem.

Theorem 2.2. Let $f(z)\in A$, and $\alpha(1/2<\alpha<1)$ be a positive real number. Suppose

$$\frac{zf'(z)}{f(z)}\neq\alpha \quad (15)$$

and

$$Re(1+\frac{zf''(z)}{f'(z)})>Re(\frac{zf'(z)}{f(z)})-\frac{1-\alpha}{2\alpha}.$$

Then we have $f(z)\in S^*(\alpha)$.

Proof. Let

$$p(z)=\frac{zf'(z)}{f(z)}, \quad (17)$$

then $p(z)$ is analytic in U and $p(0)=1$. Suppose that there exists a point $z_0\in U$ which satisfies the conditions (12) and (13) of Lemma 2.2.

Now using (17), it follows that

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{zp'(z)}{p(z)}. \quad (18)$$

Since the function p(z) and the point z_0 satisfy all conditions Lemma 2.2, therefore in view of (14) and (18) gives

$$Re(1+\frac{z_0f''(z_0)}{f'(z_0)})=Re(\frac{zp'(z_0)}{p(z_0)}+p(z_0)).$$

This is a contradiction and therefore proof of the Theorem 2.2 is completed.

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