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THE METHOD OF DETERMINING THE FUNCTION OF TRAFFIC COSTS FOR THE NODE WITH UNREGULATED FLOWS OF REQUESTS

Naumova N.A.[1]

[1]*Applied Mathematics Department, Kuban State Technological University, Krasnodar, Russia*

ABSTRACT

Modeling of transportation networks with the purpose of optimization is a vital problem. The difficulties of numerical solution of optimization problems for networks mainly depend on the analytical definition of the function of traffic costs. We provide a developed mathematical model of transportation network based on the generalized Erlang time distribution. We also propose a classification of nodes. For the case of the node with unregulated intersection of multichannel lines as a system of mass service, we obtain an analytical realization of the function of traffic costs. We describe a method of determining the parameters of the generalized Erlang law from experimental data.

Keywords: Mathematical model, Network, node, Unregulated intersection, Generalized erlang law, Function of traffic costs.

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1. INTRODUCTION

A transportation network is one of the vivid examples of network operation. Mathematical models applied for analysis of transportation networks vary according to the problems solved, mathematical apparatus, data used, and specification of traffic description [1-5]. The first macroscopic model was suggested by M. Lighthill and G. Whitham in the middle of the last century [6]. At that time there also appeared the first microscopic models ('follow-the-leader' theory) which explicitly derived an equation of motion for each individual vehicle (A. Reshel, L. Pipes, D. Gazis and others) [7, 8].

Frank A. Haigt was the first to establish the mathematical investigation of traffic flow as a separate section of applied mathematics [9]. At present there is voluminous literature on the subject. The problem of efficient management of transportation networks is, however, still topical.

At present the problems of rational operation of existing transportation networks in the centers of population as well as those of planning new ones in housing developments are undisputedly very essential. There are a lot of macro-models and micro-models of network flows distribution.

The problems of macro-modeling are aimed at searching the equilibrium distribution of flows while micro-modeling solves the problems of traffic capacity of local sections of networks. Hypotheses underlying macro-models are of different character from those of micro-models, and the problem of information exchange between the models have been not solved theoretically, neither in the form of software. Modeling and research of traffic flows often employ the theory of competitive noncooperative equilibrium which describes quite an adequate mechanism of operation of urban transportation networks [8]. Such models allow us to obtain forecasts of congestion of transportation network components. They are one of the tools of determining the efficiency of projects of the transportation network reorganizing.

The problem of flow equilibrium resolves itself into routing the traffic in the network in an optimal manner minimizing the traffic costs. The difficulty of numerical solution of such problems substantially depends on the analytical definition of the function of traffic costs.

Development of a model of network operation that will make it possible to adequately forecast the efficiency of network flow distribution from minimal initial data seems to be topical.

2. GRAPH REPRESENTATION OF THE NETWORK

Let us define the basic notions we use in this article.

We will refer to the network flows as 'non-conflict' if they are not crossed in the given sector of the network, and as 'conflict' otherwise. We will consider the node-points – the points of sources or consumption of information and those of conflict flows crossing – to be the vertices of a graph. The node-points are formed at the intersections of multichannel lines.

In our previous works we gave the following classification of node-points (NP).

A number of flows (the main ones) are freely passing the NP. The customers of the rest of flows (the secondary ones) are waiting for sufficient time intervals between arrivals of main flows for their turn to cross the NP. We will call such a node-point a 'type 1 node' or 'unregulated intersection of flows'.

Now let us consider a node-point (NP) at which traffic is alternately blocked for one of the non conflict flow groups for a fixed time to enable the crossing of the NP. We call such a node-point a 'type 2 node' or 'regulated intersection of flows' [10, 11].

We will consider a network in a traditional form of oriented graph [12]. A network is a graph each arc of which is assigned to a certain number. A flow in the graph is a group of homogeneous objects (requests) sent from one node to another. Therefore, a flow is a certain function prescribed for the graph arcs. In the developed model we show a flow in the graph as a function of density of arrival distribution (arrival times of successive service requests). Earlier we considered a model of network operation based on the Erlang time distribution for each flow. In this paper we extend the application of the model to the case when time intervals are distributed according to the generalized Erlang law. A proper selection of parameters by the generalized Erlang law will help approximate almost any distribution of a random variable.

3. METHOD OF DETERMINING THE FUNCTION OF TRAFFIC COSTS IN THE NETWORK FOR THE NODE WITH UNREGULATED FLOWS OF REQUESTS

3.1. Generalized Erlang Distribution of a Random Variable

For the generalized Erlang distribution the time interval between two requests in succession has k stages $T_0, T_1, \ldots, T_{k-1}$, the duration of which has exponential distribution with parameters $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ correspondently [13]. The Laplace transform of the function of density distribution $f_k(t)$ holds:

$$f^{(*)} = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{(s+\lambda_0)(s+\lambda_1) \cdot \ldots \cdot (s+\lambda_{k-1})}.$$

If all parameters λ_i are different, the function of distribution of the generalized Erlang law holds:

$$f_k(t) = (-1)^{k-1} \cdot \prod_{i=0}^{k-1} \lambda_i \cdot \sum_{i=0}^{k-1} \frac{e^{-\lambda_i t}}{\prod_{\substack{n=0 \\ n \neq j}}^{k-1} (\lambda_j - \lambda_n)}.$$

Or in a simple form:

$$f_k(t) = \sum_{i=0}^{k-1} a_i \lambda_i e^{-\lambda_i t},$$

if we introduce the following notation: $a_i = \prod_{\substack{n=0 \\ n \neq i}}^{k-1} \frac{\lambda_n}{\lambda_n - \lambda_i}$, in which connection $\sum_{i=0}^{k-1} a_i = 1$.

The integral distribution function: $F_k(t) = 1 - \sum_{i=0}^{k-1} a_i e^{-\lambda_i t}$.

The mathematical expectation for the generalized Erlang law can be obtained subject to the definition of Erlang flow:

$$M(T)=M\left(\sum_{i=0}^{k-1}T_i\right)=\sum_{i=0}^{k-1}\frac{1}{\lambda_i}\ .$$

The variance for generalized Erlang can be obtain with the definition of Erlang flow:

$$D(T)=D\left(\sum_{i=0}^{k-1}T_i\right)=\sum_{i=0}^{k-1}\frac{1}{(\lambda_i)^2}\ .$$

The n-th initial moment: $\nu_n=M(T^n)=\left(\sum_i a_i\lambda_i\cdot\frac{n!}{\lambda_i^{n+1}}\right)$.

3.2. Calculation of the Mean Service Time Value in the Type 1 Node for the True Generalized Erlang Distribution Hypothesis

Let a secondary flow customer has to cross L main flows in the type 1 node. Assume that time intervals are distributed by generalized Erlang of $k_1, k_2, \ldots, k_L$ orders with parameters $\{\lambda_{01}, \lambda_{11}, \ldots, \lambda_{k_1-1,1}\};\{\lambda_{02}, \lambda_{12}, \ldots, \lambda_{k_2-1,1}\};\ldots;\{\lambda_{0L}, \lambda_{1L}, \ldots, \lambda_{k_L-1,1}\}$ correspondently. Take into account that the customer arrives at the node at a random moment irrespectively of other flows requests. Let T_0 denote the minimal necessary time interval between the requests in succession in a conflict flow to continue motion.

Let t denote the random time and let T^* denote the interval between two random consecutive events which includes the random point t. $T^* = Q + R$, where R = time left before the next event, and Q = time passed after the arrival of the previous request.

Then, in accordance with the theory of random processes [11], time Q passed after the arrival of the previous request and time R remaining before the next arrival are both less than the certain preset value T_0, the probability of which can be expressed as follows:

$$P(Q<T_0)=P(R<T_0)=\int_{-\infty}^{T_0}\frac{1-F^{(k)}(t)}{m_t^{(k)}}dt=\frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\int_0^{T_0}\left(\sum_{i=0}^{k-1}a_ie^{-\lambda_it}\right)dt=$$

$$=\frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\cdot\left(\sum_{i=0}^{k-1}a_i\cdot\frac{1}{-\lambda_i}\cdot e^{-\lambda_it}\right)\Bigg|_0^{T_0}=\frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\cdot\left(\sum_{i=0}^{k-1}a_i\cdot\frac{1}{\lambda_i}\left(\cdot1-e^{-\lambda_iT_0}\right)\right)=$$

$$=\frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\cdot\left(\sum_{i=0}^{k-1}a_i\cdot\frac{1}{\lambda_i}-\sum_{i=0}^{k-1}a_i\cdot\frac{1}{\lambda_i}e^{-\lambda_iT_0}\right)=1-\frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\cdot\sum_{i=0}^{k-1}a_i\cdot\frac{1}{\lambda_i}e^{-\lambda_iT_0}$$

Consequently, the probability that no requests will arrive at the queue within time T_0 , i.e. $P(R > T_0)$:

$$P(Q > T_0) = P(R > T_0) = 1 - P(R < T_0) = \frac{1}{\sum_{i=0}^{k-1} \frac{1}{\lambda_0}} \cdot \sum_{i=0}^{k-1} a_i \cdot \frac{1}{\lambda_i} e^{-\lambda_i T_0} .$$

The numerical characteristics of random values Q and R:

the mathematical expectation: $$M(Q) = M(R) = \frac{M(T^2)}{2M(T)} = \frac{\left(\sum_i a_i \lambda_i \cdot \frac{2}{\lambda_i^3}\right)}{2 \cdot \left(\sum_{i=0}^{k-1} \frac{1}{\lambda_0}\right)};$$

$$D(Q) = D(R) = \frac{\nu_3(T)}{3M(T)} - \frac{1}{4} \frac{\left(M(T^2)\right)^2}{\left(M(T)\right)^2} = \frac{\left(\sum_i a_i \lambda_i \cdot \frac{6}{\lambda_i^4}\right)}{3 \cdot \left(\sum_{i=0}^{k-1} \frac{1}{\lambda_0}\right)} - \frac{\left(\sum_i a_i \lambda_i \cdot \frac{2}{\lambda_i^3}\right)^2}{4 \cdot \left(\sum_{i=0}^{k-1} \frac{1}{\lambda_0}\right)^2}$$

Theorem 1. Let a secondary flow customer want to cross L flows of the main direction at the type 1 node; let $\lambda_{0j}, \lambda_{1j}, \ldots, \lambda_{k_j-1}, k_j$ denote the generalized Erlang parameters for the j-th crossed flow and $j = 1, 2, \ldots, L$; T_0 denote the minimal time interval between the requests in succession in a conflict flow to continue motion. Then for the first customer in the secondary flow the service time equals:

$$m_z = \frac{\left(\sum_i a_i \lambda_i \cdot \frac{2}{\lambda_i^3}\right)}{2 \cdot \left(\sum_{i=0}^{k-1} \frac{1}{\lambda_0}\right)} \cdot \Phi_0(T_0) + \frac{1}{\sum_{i=0}^{k-1} \frac{1}{\lambda_0}} \Phi_0(T_0) \cdot \Phi(T_0) \cdot \frac{1}{\left(1 - \Phi(T_0)\right)},$$

where $$\Phi_0(T_0) = 1 - \prod_{j=1}^{L} \left(1 - \frac{1}{\sum_{i=0}^{k_j-1} \frac{1}{\lambda_{0j}}} \cdot \sum_{i=0}^{k_j-1} a_{ij} \cdot \frac{1}{\lambda_{ij}} e^{-\lambda_{ij} T_0} \right),$$

$$\Phi(T_0)=1-\left(1-\sum_{i=0}^{k_1-1}a_{i1}e^{-\lambda_{i1}T_0}\right)\prod_{j=2}^{L}\left(1-\frac{1}{\sum_{i=0}^{k_j-1}\frac{1}{\lambda_{0j}}}\cdot\sum_{i=0}^{k_j-1}a_{ij}\cdot\frac{1}{\lambda_{ij}}e^{-\lambda_{ij}T_0}\right).$$

Proof

Let us consider value $Z=\sum_{i=1}^{X}T_i$, where $T_1=R_1$ which is the time remaining till the arrival of the next request-in-queue in the first crossed flow of the main direction (Flow (1)); $T_i, i=2,3,4,...$ is the interval between arrivals of the *(i – 1)* -th and *i*-th requests in Flow (1); X is the integral random value which is equal to the number of requests to be serviced before the motion can be continued (it is evident that $X=m$, when first *m* intervals are less than the necessary time and the next $(m+1)$-th interval is already bigger than T_0).

Let us introduce the following random values:

$Y_0=\min\limits_{j=2..L}\{R_1, R_1^j\}$; $Y_i=\min\limits_{j=2..L}\{T_{i+1}, R_{i+1}^j\}=Y$, *i=1, 2, 3, …*

Here R_{i+1}^j = time interval in the j-th flow before the arrival of the $(i+1)$-th request in Flow (1).

the function of distribution of random value Y_j $j=1,2,3,...$ holds [11]:

$$\Phi(y)=1-\prod_{i=1}^{L}\left(1-F_i(y)\right)=P(Y_j<y)$$

where $F_i(y)$ = function of distribution of the *i*-th random value.

Therefore, the function of distribution of random value $Y_0=\min\limits_{j=2..L}\{R_1, R_1^j\}$:

$$\Phi_0(y)=P(Y_0<y)=1-\prod_{j=1}^{L}\left(1-\frac{1}{\sum_{i=0}^{k_j-1}\frac{1}{\lambda_{0j}}}\cdot\sum_{i=0}^{k_j-1}a_{ij}\cdot\frac{1}{\lambda_{ij}}e^{-\lambda_{ij}y}\right).$$

The functions of distribution of random values $Y_i=\min\limits_{j=2..L}\{T_{i+1}, R_{i+1}^j\}=Y$, *i=1, 2, 3, …*

coincide and take the form:

$$\Phi(y)=P(Y<y)=1-\left(1-\sum_{i=0}^{k_1-1}a_{i1}e^{-\lambda_{i1}y}\right)\prod_{j=2}^{L}\left(1-\frac{1}{\sum_{i=0}^{k_j-1}\frac{1}{\lambda_{0j}}}\cdot\sum_{i=0}^{k_j-1}a_{ij}\cdot\frac{1}{\lambda_{ij}}e^{-\lambda_{ij}y}\right),$$

where $\lambda_{01},\lambda_{11},\ldots,\lambda_{k_1-1},k_1=$ parameters of the generalized Erlang law for the first crossed flow;

$\lambda_{0j},\lambda_{1j},\ldots,\lambda_{k_j-1},k_j$ = parameters of the generalized Erlang law for the j-th crossed flow.

Therefore,

$$m_Z=M(Z)=\sum_{m=1}^{\infty}M(Z/X=m)\cdot P_m\ ;$$

$$P_0=P(Y_0>T_0);$$
$$P_1=P(Y_0<T_0)\cdot P(Y_m>T_0));$$
$$P_m=P(Y_0<T_0)\cdot P(Y_{m-1}<T_0)^{m-1}P(Y_m>T_0)),\qquad m\geq 2$$

That is $P_1=P(X=1)=P(Y_0<T_0)\cdot(1-\Phi(T_0))$;

$P_m=P(X=m)=P(Y_0<T_0)\cdot(\Phi(T_0))^{m-1}\cdot(1-\Phi(T_0))$ with $m\geq 2$;

Then $P_0=1-\Phi_0(T_0)=\prod_{j=1}^{L}\left(1-\frac{1}{\sum_{i=0}^{k_j-1}\frac{1}{\lambda_{0j}}}\cdot\sum_{i=0}^{k_j-1}a_{ij}\cdot\frac{1}{\lambda_{ij}}e^{-\lambda_{ij}T_0}\right)$.

Proceed with calculations:

$$m_z = M(Z) = \sum_{m=1}^{\infty} M(Z / X = m)\cdot P_m = \sum_{m=1}^{\infty}\left(\frac{\left(\sum_i a_i\lambda_i\cdot\frac{2}{\lambda_i^3}\right)}{2\cdot\left(\sum_{i=0}^{k-1}\frac{1}{\lambda_0}\right)} + (m-1)\frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\right)P_m =$$

$$= \frac{\left(\sum_i a_i\lambda_i\cdot\frac{2}{\lambda_i^3}\right)}{2\cdot\left(\sum_{i=0}^{k-1}\frac{1}{\lambda_0}\right)}\sum_{m=1}^{\infty} P_m + \frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\sum_{m=1}^{\infty}(m-1)P_m =$$

$$= \frac{\left(\sum_i a_i\lambda_i\cdot\frac{2}{\lambda_i^3}\right)}{2\cdot\left(\sum_{i=0}^{k-1}\frac{1}{\lambda_0}\right)}\sum_{m=1}^{\infty}P(Y_0 < T_0)\cdot(\Phi(T_0))^{m-1}\cdot(1-\Phi(T_0)) +$$

$$+ \frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\sum_{m=2}^{\infty}(m-1)P(Y_0 < T_0)\cdot(\Phi(T_0))^{m-1}\cdot(1-\Phi(T_0)),$$

$$m_z = \frac{\left(\sum_i a_i\lambda_i\cdot\frac{2}{\lambda_i^3}\right)}{2\cdot\left(\sum_{i=0}^{k-1}\frac{1}{\lambda_0}\right)}\cdot P(Y_0 < T_0)\cdot(1-\Phi(T_0))\cdot\frac{1}{1-\Phi(T_0)} +$$

$$+ \frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}P(Y_0 < T_0)\cdot(1-\Phi(T_0))\sum_{m=2}^{\infty}(m-1)\cdot(\Phi(T_0))^{m-1} =$$

$$= \frac{\left(\sum_i a_i\lambda_i\cdot\frac{2}{\lambda_i^3}\right)}{2\cdot\left(\sum_{i=0}^{k-1}\frac{1}{\lambda_0}\right)}\cdot P(Y_0 < T_0) + \frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}P(Y_0 < T_0)\cdot(1-\Phi(T_0))\cdot\Phi(T_0)\cdot\left(\sum_{m=2}^{\infty}(\Phi(T_0))^{m-1}\right)^{/} =$$

$$= \frac{\left(\sum_i a_i\lambda_i\cdot\frac{2}{\lambda_i^3}\right)}{2\cdot\left(\sum_{i=0}^{k-1}\frac{1}{\lambda_0}\right)}\cdot P(Y_0 < T_0) + \frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}P(Y_0 < T_0)\cdot(1-\Phi(T_0))\cdot\Phi(T_0)\cdot\frac{1}{(1-\Phi(T_0))^2}$$

$$m_z = \frac{\left(\sum_i a_i\lambda_i\cdot\frac{2}{\lambda_i^3}\right)}{2\cdot\left(\sum_{i=0}^{k-1}\frac{1}{\lambda_0}\right)}\cdot\Phi_0(T_0) + \frac{1}{\sum_{i=0}^{k-1}\frac{1}{\lambda_0}}\Phi_0(T_0)\cdot\Phi(T_0)\cdot\frac{1}{(1-\Phi(T_0))}.$$

The theorem is proven.

A type 1 node can be represented as a system of mass service with an unlimited queue. The flow of requests is a flow of requests of a secondary flow arriving at the node; service time is the portion of time that a request will spend in the queue waiting for the opportunity to continue motion. In our previous works [14-16] we proved the following theorem with the help of method of pseudostates.

Theorem 2. Let the service time is distributed by exponential law with parameter μ; the flow of requests is distributed by generalized Erlang with parameters $\lambda_0, \lambda_1, ..., \lambda_{k-1}, k$; no more than *n* requests can be serviced at any one time. Then, for a stationary process probabilities of states of the mass service system with an unlimited queue length are expressed by the formulas:

$$p_m = \frac{\left(\frac{\lambda_0}{\mu}\right)^m}{m! \cdot b\left(\sum_{j=0}^{n}\frac{\alpha^j}{j!} + \frac{\alpha^n}{n!}\frac{\alpha/n}{1-\frac{\alpha}{n}}\right)} \quad (m = 0, 1, \ldots, n-1);$$

$$p_{n+i} = \frac{\alpha^{n+i}}{n! n^i \cdot b\left(\sum_{j=0}^{n}\frac{\alpha^j}{j!} + \frac{\alpha^n}{n!}\frac{\alpha/n}{1-\frac{\alpha}{n}}\right)} \quad (i = 0, 1, 2, \ldots).,$$

where $b = \sum_{i=0}^{k}\frac{\lambda_0}{\lambda_i} > 1,\ \alpha = \frac{\lambda_0}{k \cdot \mu}$.

Subject to Theorem 2 we obtained the mean number of requests queued or being serviced:

$$M(l) = bp_0\left(\sum_{j=1}^{n}\frac{\alpha^j}{(j-1)!} + \frac{\alpha^{n+1}}{n!}\frac{1}{1-\frac{\alpha}{n}} + \frac{\alpha^{n+1}}{n!\, n\cdot\left(1-\alpha/n\right)^2}\right).$$

For the type 1 node the number of servers $n = 1$, then

$$M(l) = bp_0\left(\alpha + \frac{\alpha^2(2-\alpha)}{(1-\alpha)^2}\right), \quad \text{where} \quad p_0 = 1\Big/ b\left(1+\alpha+\frac{\alpha^2}{1-\alpha}\right).$$

On simplification, $M(l) = \frac{\alpha}{1-\alpha}$.

Then an average delay of the secondary flow requests at the type 1 node is:

$$W_H = m_Z \cdot M(l) = \frac{\alpha \cdot m_z}{1-\alpha},$$

where $\alpha = \frac{\lambda_0}{k \cdot \mu}$, $\mu = 1/(m_z)$.

3.3. Function of Traffic Costs for a Type 1 Node

According to the theory of flow equilibrium [8], in order to obtain the numerical values of equilibrium distribution it is necessary first to solve the problem of construction of the function of traffic costs. The prevalent assumption on the properties of the function of traffic costs is a logical assumption on its additive dependence upon the traffic costs of passing particular arcs and vertices.

According to the purpose of optimization a function of traffic costs for the node can be chosen:

1) $\vec{\mu}(z_n)$ - the weight of vertex z_n (a node-point) for a flow of the given direction;

2) $\mu(z_n)$ - the integrated weight of vertex z_n (a node-point);

3) $\omega_M(z_n)$- the mean delay of requests of the chosen directions.

For a type 1 node:

1) $\vec{\mu}(z_n) = \sum_{i \in M} \frac{W_{Hi}}{\sum_{j=0}^{k_i-1} \frac{1}{\lambda_{ji}}}$, where M is the set of the chosen directions;

2) $\mu(z_n) = \sum_{i \in \Omega} \frac{W_{Hi}}{\sum_{j=0}^{k_i-1} \frac{1}{\lambda_{ji}}}$, where Ω is the set of all directions of request flows through the given node;

3) $\omega_M(z_n) = \sum_{i \in M}\left(\frac{W_{Hi}}{\sum_{j=0}^{k_i-1} \frac{1}{\lambda_{ji}}}\right) \Big/ \sum_{i \in M}\left(\sum_{j=0}^{k_i-1} \frac{1}{\lambda_{ji}}\right)$, where M is the set of the chosen directions.

We developed Delpi-based computer programs, defining the form of the function of traffic costs in the node depending on the parameters of the generalized Erlang law.

Let Ψ be the set of modes of flow distribution for the given node of the graph. The optimal flow distribution at the node is the solution of the problem (according to the aim):

1) $\vec{\mu}(z_n)\_opt = \min_{\Psi}\{\vec{\mu}(z_n)\}$;

2) $\mu(z_n)\_opt = \min_{\Psi}\{\mu(z_n)\}$;

3) $\omega_M(z_n)\_opt = \min_{\Psi}\{\omega_M(z_n)\}$.

4. DEFINITION OF PARAMETERS OF THE GENERALIZED ERLANG LAW FROM EXPERIMENTAL DATA

The above model applied, the parameters of the generalized Erlang law can be defined with the method of moments, i. e. by equating theoretical and empirical values of the mathematical expectation and variance.

Round off the value of number $k^* = \frac{\bar{x}_B^2}{\hat{s}^2}$, then choose parameter k.

With $k = 2$.

$$\begin{cases} \frac{1}{\lambda_0} + \frac{1}{\lambda_0} = \bar{x}_B \\ \frac{1}{(\lambda_0)^2} + \frac{1}{(\lambda_1)^2} = (\sigma_B)^2 \end{cases}$$

Then, if $(\sigma_B)^2 < \left(\bar{x}_B\right)^2 < 2(\sigma_B)^2$ the values of the parameters are:

$$\lambda_0 = \frac{2}{\bar{x}_B + \sqrt{2(\sigma_B)^2 - (x_B)^2}}, \quad \lambda_1 = \frac{2}{\bar{x}_B - \sqrt{2(\sigma_B)^2 - (x_B)^2}}.$$

With $k = 3$.

$$\begin{cases} \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \bar{x}_B \\ \frac{1}{(\lambda_0)^2} + \frac{1}{(\lambda_1)^2} + \frac{1}{(\lambda_2)^2} = (\sigma_B)^2 \end{cases}$$

Let $\lambda_1 = x \cdot \lambda_0, \lambda_2 = x \cdot \lambda_1 = x^2 \cdot \lambda_0$, then

$$x = \frac{\left((\bar{x}_B)^2 + (\sigma_B)^2\right) + \sqrt{\left(-(\bar{x}_B)^2 + 3(\sigma_B)^2\right) \cdot \left(3(\bar{x}_B)^2 - (\sigma_B)^2\right)}}{2\left((\bar{x}_B)^2 - (\sigma_B)^2\right)},$$

with $(\sigma_B)^2 < \left(\bar{x}_B\right)^2 < 3(\sigma_B)^2$.

After that, we calculate parameter λ_0:

$$\lambda_0 = \frac{x^2 + x + 1}{x^2} \cdot \frac{1}{\bar{x}_B}.$$

With $k = 4$.

$$\begin{cases} \frac{1}{\lambda_0}+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}=\bar{x}_B \\ \frac{1}{(\lambda_0)^2}+\frac{1}{(\lambda_1)^2}+\frac{1}{(\lambda_2)^2}+\frac{1}{(\lambda_3)^2}=(\sigma_B)^2 \end{cases}$$

Let $\lambda_1 = x\cdot\lambda_0,\ \lambda_2 = x\cdot\lambda_1 = x^2\cdot\lambda_0,\ \lambda_3 = x\cdot\lambda_2 = x^3\cdot\lambda_0$.

Then, if $(\sigma_B)^2 < \left(\bar{x}_B\right)^2$ the values of the parameters become as follows:

$$x=\frac{y\pm\sqrt{y^2-4}}{2},\ \text{where}\ y=\frac{(\sigma_B)^2+\sqrt{\left((\bar{x}_B)^2-(\sigma_B)^2\right)^2+(\bar{x}_B)^4}}{(\bar{x}_B)^2-(\sigma_B)^2};$$

$$\lambda_0=\frac{(x^2+1)(x+1)}{x^3}\cdot\frac{1}{\bar{x}_B}.$$

Note that if $k^*=\frac{\bar{x}_B^2}{\hat{s}^2}$ is a whole number, then for all $k\in\{2,3,4\}$ the value $x=1$, and, therefore, all λ_i coincide. Thus, we have a special Erlang distribution which was described in detail in in our other works [17, 18].

5. CONCLUSIONS

The above results is our generalized research on optimization of traffic flow distribution in the network [14-16]. The hypothesis on Erlang distribution of time intervals between the requests allowed us to develop the mathematical model providing satisfactory accuracy of estimation of the results of the network efficiency. Besides, the minimal number of initial parameters made less expensive the development of the database for evaluation of quality of reorganization within the network. The generalized Erlang law will allow for A proper selection of parameters by the generalized Erlang law will allow one to approximate almost any distribution with the sufficient accuracy, and, therefore, will enable to extend the application of the model to the flows of higher density passing through a number of nodes. In its turn, this will allow for a greater accuracy when solving optimization problems in the theory of traffic flow.

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