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ALTERNATIVE WAY OF STATISTICAL DATA ANALYSIS: L-MOMENTS AND TL-MOMENTS OF PROBABILITY DISTRIBUTION

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ABSTRACT

This paper deals with an alternative method to characterize some probability distribution, i.e. L-moments and TL-moments. Moments and cumulants were commonly used for this purpose before. Moment method of parameter estimation is indeed very simple, but it is very inaccurate, especially if there are the moments of the third or higher order. These problems mainly occur in the case of small samples. Maximum likelihood method has been considered as the most accurate method of parameter estimation for a long time. Using the method of L-momednts gives more accurate results than using the maximum likelihood method in many cases when estimating parameters of probability distribution. The method of TL-moments brings still more accurate results than the method of L-moments. This paper deals with these three methods of parameter estimation and with the comparison of their accuracy. Three-parametric lognormal curves constitute basic probability distribution. They were used as the model distribution of income in the Czech Repubublic in 1992, 1996, 2002 and 2004–2007.

Keywords: L-moments and TL-moments of probability distribution, Sample L-moments and TL-moments, Probability density function, Distribution function, Quantile function, Order statistics, Income distribution.

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Contribution/ Originality

This study contributes in the existing literature and it develops the theory of estimation of parameters of continuous probability distributions in the case of large economic data sets. This study originates some new formulas, proofs and derivations, which were not published at all. This study is one of very few studies which have investigated the use of the method of TL-moments on economic data.

1. INTRODUCTION

Application of L-moments and TL-moments includes all, i.e. statistical description of sample data set, modeling sample distribution using some theoretical probability distribution with parameters estimated with the method of L-moments or TL-moments and parametric hypothesis testing. The theory of L-moments and TL-moments includes the established methods such as the use of order statistics and the Gini mean difference. It leads to some promising innovations in the area of measuring skewness and kurtosis of the distribution and provides relatively new methods of parameter estimation for an individual distribution. The main advantage of L-moments and TL-moments is that they exist for any random variable for which the expected value can be defined. They provide more accurate results especially for small samples. L-moments and TL-moments sometimes bring even more efficient parameter estimations of the parametric distribution than those estimated by the maximum likelihood method for small samples in particular, see [1]. They are also more resistant and less prone to estimation bias, approximation by the asymptotic normal distribution being more accurate in finite samples, see [2]. The statement of this research consists in showing the advantages of the method of L-moments ant its modification the method of TL-moments in terms of the accuracy of parameter point estimation of the continuous distributions. Microsoft Excel Spreadsheet program and program packet R were used for the calculations in the framework of this research. Let X be a random variable being distributed with the distribution function $F(x)$ and quantile function $x(F)$ and let $X_1, X_2, \ldots, X_n$ be a random sample of the sample size n from this distribution. Then $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ are order statistics of the random sample of the sample size n which comes from the distribution of the random variable X.

2. THEORY

2.1. L-Moments and Sample L-Moments

The issue of L-moments is discussed, for example, in Adamowski [3] or [3]. We consider continuous random variable X with distribution function $F(x)$ and with quantile function $x(F)$. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be order statistics of a random sample of the sample size n which comes from the distribution of the random variable X. The r-th L-moment of this variable X can be defined

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r-j:r}), \quad r = 1, 2, \ldots . \tag{1}$$

Now we can define the expected value of the r-th order statistic, which is related to the random sample with sample size n

$$E(X_{r:n}) = \frac{n!}{(r-1)! \cdot (n-r)!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-1} \cdot [1-F(x)]^{n-r} \mathrm{d}\, F(x). \tag{2}$$

After the substitution the formula (2) into (1), we get

$$\lambda_r = \int_0^1 x(F) \cdot P^*_{r-1}[F(x)]\, \mathrm{d}\, F(x), \quad r = 1, 2, \dots, \tag{3}$$

where

$$P^*_r[F(x)] = \sum_{j=0}^{r} p^*_{r,j} \cdot [F(x)]^j \quad \text{a} \quad p^*_{r,j} = (-1)^{r-j} \cdot \binom{r}{j} \cdot \binom{r+j}{j}, \tag{4}$$

where $P^*_r[F(x)]$ is the r-th shifted Legendre polynomial. Substituting equation (2) into (1), we can also observe

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{r!}{(r-j-1)! \cdot j!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-j-1} \cdot [1 - F(x)]^j\, \mathrm{d}\, F(x), \quad r = 1, 2, \dots. \tag{5}$$

The letter “L” in “L-moments” indicates that the r-th L-moment λ_r is a linear function of the expected value of a certain linear combination of order statistics. The estimate of the r-th L-moment λ_r, based on the sample, is thus the linear combination of order data values, i.e. L-statistics. Now we obtain the first four L-moments of probability distribution

$$\lambda_1 = E(X_{1:1}) = \int_0^1 x(F)\, \mathrm{d}\, F(x), \tag{6}$$

$$\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) = \int_0^1 x(F) \cdot [2F(x) - 1]\, \mathrm{d}\, F(x), \tag{7}$$

$$\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) = \int_0^1 x(F) \cdot \{6[F(x)]^2 - 6F(x) + 1\}\, \mathrm{d}\, F(x), \tag{8}$$

$$\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) = \int_0^1 x(F) \cdot \{20[F(x)]^3 - 30[F(x)]^2 + 12[F(x)] - 1\}\, \mathrm{d}\, F(x). \tag{9}$$

Even if some of the conventional moments of the probability distribution do not exist, it can be described by its L-moments (this is not the case of lognormal curves). However, the opposite does not apply. It can be proved that the first L-moment λ_1 is a location characteristic, the second L-moment λ_2 being a variability characteristic. It is often desirable to standardize higher L-moments λ_r, $r \geq 3$, so that they can be independent of specific units of the random variable X. We can obtain the ratio of L-moments of the r-th order of the random variable X

$$\tau_r = \frac{\lambda_r}{\lambda_2}, \quad r = 3, 4, \dots. \tag{10}$$

The analogy of classical variation coefficient, i.e. L-variation coefficient has the form

$$\tau = \frac{\lambda_2}{\lambda_1}. \tag{11}$$

Table-1. Formulas for distribution or quantile functions, L-moments and their ratios for lognormal and generalized Pareto probability distributions

Distribution	Distribution function $F(x)$ or quantile function $x(F)$	L-moments and ratios of L-moments
Lognormal	$F(x) = \Phi\left\{\dfrac{\ln[x(F) - \xi] - \mu}{\sigma}\right\}$	$\lambda_1 = \xi + \exp\left(\mu + \dfrac{\sigma^2}{2}\right)$ $\lambda_2 = \exp\left(\mu + \dfrac{\sigma^2}{2}\right) \cdot \operatorname{erf}\left(\dfrac{\sigma}{2}\right)$ $\tau_3 = 6\pi^{-\frac{1}{2}} \cdot \dfrac{\int_0^{\frac{\sigma}{2}} \operatorname{erf}\left(\frac{x}{\sqrt{3}}\right) \cdot \exp(-x^2)\, \mathrm{d}x}{\operatorname{erf}\left(\frac{\sigma}{2}\right)}$
Generalized Pareto	$x(F) = \xi + \alpha \cdot \dfrac{1 - [1 - F(x)]^k}{k}$	$\lambda_1 = \xi + \dfrac{\alpha}{1 + k}$ $\lambda_2 = \dfrac{\alpha}{(1 + k) \cdot (2 + k)}$ $\tau_3 = \dfrac{1 - k}{3 + k}$ $\tau_4 = \dfrac{(1 - k) \cdot (2 - k)}{(3 + k) \cdot (4 + k)}$

Source: Hosking [1]; own calculations

The ratio of L-moments τ_3 is a skewness characteristic, the ratio of L-moments τ_4 being a kurtosis characteristic of the corresponding probability distribution. Main properties of the probability distribution are very well summarized by the following four characteristics: L-location λ_1, L-variability λ_2, L-skewness τ_3 and L-kurtosis τ_4. L-moments λ_1 and λ_2, the L-coefficient of variation τ and ratios of L-moments τ_3 and τ_4 are the most useful characteristics for the summarization of the probability distribution. Their main properties are existence (if the expected value of the distribution is finite, then all its L-moments exist) and uniqueness (if the expected value of the distribution is finite, then L-moments define the only distribution).

Using equations (6)–(9) and (10), we obtain both the expressions for L-moments and L-moments ratios for lognormal and generalized Pareto probability distributions, see Table 1.

L-moments are mostly calculated using some random sample, which is got from an unknown probability distribution. Since the r-th L-moment λ_r is the function of the expected values of order statistics of a random sample of the sample size r, it is natural to estimate it using the so-called U-statistic. That is an appropriate function of sample order statistics.

Let $x_1, x_2, \ldots, x_n$ be the sample and $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ the ordered sample. Now the r-th sample L-moment has the form

$$l_r = \binom{n}{r}^{-1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} \sum \ldots \sum \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot x_{i_{r-j}:n}, \quad r = 1, 2, \ldots, n. \tag{12}$$

Hence we have the first four sample L-moments

$$l_1 = \frac{1}{n} \cdot \sum_i x_i, \tag{13}$$

$$l_2 = \frac{1}{2} \cdot \binom{n}{2}^{-1} \cdot \sum_{i>j}\sum (x_{i:n} - x_{j:n}), \tag{14}$$

$$l_3 = \frac{1}{3} \cdot \binom{n}{3}^{-1} \cdot \sum_{i>j>k}\sum\sum (x_{i:n} - 2x_{j:n} + x_{k:n}), \tag{15}$$

$$l_4 = \frac{1}{4} \cdot \binom{n}{4}^{-1} \cdot \sum_{i>j>k>l}\sum\sum\sum (x_{i:n} - 3x_{j:n} + 3x_{k:n} - x_{l:n}). \tag{16}$$

The positive properties of U-statistics are the absence of bias, asymptotic normality and a slight resistance due to the influence of outliers, see [1].

When calculating the r-th sample L-moment, it is not necessary to repeat the process over all sub-sets of the sample size r, since this statistic can be expressed directly as a linear combination of order statistics of a random sample of the sample size n.

If we assume an estimate of $E(X_{r:r})$ obtained with the use of U-statistics, it can be written as $r \cdot b_{r-1}$, where

$$b_r = \frac{1}{n} \cdot \binom{n-1}{r}^{-1} \cdot \sum_{j=r+1}^{n} \binom{j-1}{r} \cdot x_{j:n}, \tag{17}$$

namely

$$b_0 = \frac{1}{n} \cdot \sum_{j=1}^{n} x_{j:n}, \tag{18}$$

$$b_1 = \frac{1}{n} \cdot \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} \cdot x_{j:n}, \tag{19}$$

$$b_2 = \frac{1}{n} \cdot \sum_{j=3}^{n} \frac{(j-1)\cdot(j-2)}{(n-1)\cdot(n-2)} \cdot x_{j:n}, \tag{20}$$

and universally

$$b_r = \frac{1}{n} \cdot \sum_{j=r+1}^{n} \frac{(j-1)\cdot(j-2)\cdot\ldots\cdot(j-r)}{(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r)} \cdot x_{j:n}. \tag{21}$$

Thus the first sample L-moments can be written as

$$l_1 = b_0, \tag{22}$$

$$l_2 = 2b_1 - b_0, \tag{23}$$

$$l_3 = 6b_2 - 6b_1 + b_0, \tag{24}$$

$$l_4 = 20 b_3 - 30 b_2 + 12 b_1 - b_0. \tag{25}$$

We can generalize this

$$l_{r+1} = \sum_{k=0}^{r} p_{r,k}^{*} \cdot b_k, \quad r = 0, 1, \ldots, n-1, \tag{26}$$

where

$$p_{r,k}^{*} = (-1)^{r-k} \cdot \binom{r}{k} \cdot \binom{r+k}{k} = \frac{(-1)^{r-k} \cdot (r+k)!}{(k!)^2 \cdot (r-k)!}. \tag{27}$$

Sample L-moments are used in a similar way as sample conventional L-moments, summarizing the basic properties of the sample distribution, which are the location (level), variability, skewness and kurtosis. Thus, sample L-moments allow an estimation the corresponding properties of the probability distribution from which the sample originates and can be used in estimating the parameters of the relevant probability distribution. We often prefer L-moments to conventional moments within such applications, since sample L-moments – as the linear functions of sample values – are less sensitive to sample variability or measurement errors in extreme observations than conventional moments. L-moments therefore lead to more accurate and robust estimates of characteristics or parameters of the basic probability distribution.

Sample L-moments have been used previously in statistics, but not as part of a unified theory. The first sample L-moment l_1 is a sample L-location (sample average), the second sample L-moment l_2 being a sample L-variability. The natural estimation of L-moments (10) ratio is the sample ratio of L-moments

$$t_r = \frac{l_r}{l_2}, \quad r = 3, 4, \ldots. \tag{28}$$

Hence t_3 is a sample L-skewness and t_4 is a sample L-kurtosis. Sample ratios of L-moments t_3 and t_4 may be used as the characteristics of skewness and kurtosis of a sample data set.

The Gini mean difference relates both to sample L-moments, having the form of

$$G = \binom{n}{2}^{-1} \cdot \sum_{i>j}\sum (x_{i:n} - x_{j:n}), \tag{29}$$

and the Gini coefficient which depends only on a single parameter σ in the case of the two-parametric lognormal distribution, depending, however, on the values of all three parameters in the case of the three-parametric lognormal distribution. Table 2 presents the expressions for parameter estimations of lognormal and generalized Pareto probability distributions obtained using the method of L-moments. For more details see, for example, [1, 4-14].

Table-2. Formulas for parameter estimations made by the method of L-moments of lognormal and generalized Pareto probability distributions

Distribution	Parameter estimation
Lognormal	$z = \sqrt{\frac{8}{3}} \cdot \Phi^{-1}\left(\frac{1+t_3}{2}\right)$ [1)] $\hat{\sigma} = 0{,}999\,281 z - 0{,}006\,118\, z^3 + 0{,}000\,127\, z^5$ $\hat{\mu} = \ln \frac{l_2}{\mathrm{erf}\left(\frac{\sigma}{2}\right)} - \frac{\hat{\sigma}^2}{2}$ $\hat{\xi} = l_1 - \exp\left(\hat{\mu} + \frac{\hat{\sigma}^2}{2}\right)$
Generalized Pareto	(ξ known) $\hat{k} = \frac{l_1}{l_2} - 2$ $\hat{\alpha} = (1+\hat{k}) \cdot l_1$

Source: Hosking [1] own calculations

2.2. TL-Moments and Sample TL-Moments

An alternative robust version of L-moments is introduced in this subchapter. The modification is called "trimmed L-moments" and it is termed TL-moments. The expected values of order statistics of a random sample in the definition of L-moments of probability distributions are replaced with those of a larger random sample, its size growing correspondingly to the extent of the modification, as shown below. Certain advantages of TL-moments outweigh those of conventional L-moments and central moments. TL-moment of the probability distribution may exist despite the non-existence of the corresponding L-moment or central moment of this probability distribution, as it is the case of the Cauchy distribution. Sample TL-moments are more resistant to outliers in the data. The method of TL-moments is not intended to replace the existing robust methods but rather supplement them, particularly in situations when we have outliers in the data.

In this alternative robust modification of L-moments, the expected value $E(X_{r-j:r})$ is replaced with the expected value $E(X_{r+t_1-j\,:\,r+t_1+t_2})$. Thus, for each r, we increase the sample size of a random sample from the original r to $r + t_1 + t_2$, working only with the expected values of these r modified order statistics $X_{t_1+1:r+t_1+t_2}$, $X_{t_1+2:r+t_1+t_2}$, …, $X_{t_1+r:r+t_1+t_2}$ by trimming the smallest t_1 and largest t_2 from the conceptual random sample. This modification is called the r-th trimmed L-moment (TL-moment) and marked as $\lambda_r^{(t_1,t_2)}$. Thus, TL-moment of the r-th order of the random variable X is defined as

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1) $\Phi^{-1}(\cdot)$ is a quantile function of the standardized normal distribution

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r}\cdot\sum_{j=0}^{r-1}(-1)^j\cdot\binom{r-1}{j}\cdot E(X_{r+t_1-j:r+t_1+t_2}),\quad r=1,2,\ldots. \tag{30}$$

It is evident from the expressions (30) and (1) that TL-moments are reduced to L-moments, where $t_1 = t_2 = 0$. So, substituting $t_1 = t_2 = 0$ into the equation (30) and after simple adjustment we receive the relationship (1). Although we can also consider applications where the adjustment values are not equal, i.e. $t_1 \neq t_2$, we will focus here only on the symmetric case $t_1 = t_2 = t$. Then the expression (30) can be rewritten

$$\lambda_r^{(t)} = \frac{1}{r}\cdot\sum_{j=0}^{r-1}(-1)^j\cdot\binom{r-1}{j}\cdot E(X_{r+t-j:r+2t}),\quad r=1,2,\ldots. \tag{31}$$

Thus, for example, $\lambda_1^{(t)} = E(X_{1+t:1+2t})$ is the expected value of the median of the conceptual random sample of $1 + 2t$ size. It is necessary to note that $\lambda_1^{(t)}$ is equal to zero for distributions that are symmetrical around zero.

If we consider $t = 1$, we get the following first four TL-moments

$$\lambda_1^{(1)} = E(X_{2:3}), \tag{32}$$

$$\lambda_2^{(1)} = \frac{1}{2}E(X_{3:4} - X_{2:4}), \tag{33}$$

$$\lambda_3^{(1)} = \frac{1}{3}E(X_{4:5} - 2X_{3:5} + X_{2:5}), \tag{34}$$

$$\lambda_4^{(1)} = \frac{1}{4}E(X_{5:6} - 3X_{4:6} + 3X_{3:6} - X_{2:6}). \tag{35}$$

The measurements of location, variability, skewness and kurtosis of the probability distribution analogous to conventional L-moments (6)–(9) are based on $\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}$ a $\lambda_4^{(1)}$.

The expected value $E(X_{r:n})$ can be written using the formula (2). With the use of the equation (2), we can express the right side of the equation (31) again as

$$\lambda_r^{(t)} = \frac{1}{r}\cdot\sum_{j=0}^{r-1}(-1)^j\cdot\binom{r-1}{j}\cdot\frac{(r+2t)!}{(r+t-j-1)!\cdot(t+j)!}\cdot\int_0^1 x(F)\cdot[F(x)]^{r+t-j-1}\cdot[1-F(x)]^{t+j}\,\mathrm{d}F(x),\ r=1,2,\ldots. \tag{36}$$

It is necessary to point out that $\lambda_r^{(0)} = \lambda_r$ represents a normal r-th L-moment with no respective adjustments. Expressions (32)–(35) for the first four TL-moments ($t = 1$) may be written in an alternative way as

$$\lambda_1^{(1)} = 6\cdot\int_0^1 x(F)\cdot[F(x)]\cdot[1-F(x)]\,\mathrm{d}F(x), \tag{37}$$

$$\lambda_2^{(1)} = 6 \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot [2F(x) - 1] \, \mathrm{d}F(x), \tag{38}$$

$$\lambda_3^{(1)} = \frac{20}{3} \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot \{5[F(x)]^2 - 5F(x) + 1\} \, \mathrm{d}F(x), \tag{39}$$

$$\lambda_4^{(1)} = \frac{15}{2} \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot \{14[F(x)]^3 - 21[F(x)]^2 + 9[F(x)] - 1] \, \mathrm{d}F(x). \tag{40}$$

The distribution can be determined by its TL-moments, even though some of its L-moments or conventional moments do not exist. For example, $\lambda_1^{(1)}$ (the expected value of the median of a conceptual random sample of sample size three) exists for the Cauchy distribution, despite the non-existence of the first L-moment λ_1.

TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ can be defined analogously as L-skewness τ_3 and L-kurtosis τ_4

$$\tau_3^{(t)} = \frac{\lambda_3^{(t)}}{\lambda_2^{(t)}}, \tag{41}$$

$$\tau_4^{(t)} = \frac{\lambda_4^{(t)}}{\lambda_2^{(t)}}. \tag{42}$$

Let $x_1, x_2, \ldots, x_n$ be a sample and $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ an order sample. The expression

$$\hat{E}(X_{j+1:j+l+1}) = \frac{1}{\binom{n}{j+l+1}} \cdot \sum_{i=1}^{n} \binom{i-1}{j} \cdot \binom{n-i}{l} \cdot x_{i:n} \tag{43}$$

is considered to be an unbiased estimate of the expected value of the $(j+1)$-th order statistic $X_{j+1:j+l+1}$ in the conceptual random sample of sample size $(j+l+1)$. Now we will assume that in the definition of TL-moment $\lambda_r^{(t)}$ in (31), the expression $E(X_{r+t-j:r+2t})$ is replaced by its unbiased estimate

$$\hat{E}(X_{r+t-j:r+2t}) = \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \tag{44}$$

which is obtained by assigning $j \to r+t-j-1$ a $l \to t+j$ in (43). The r-th sample TL-moment has the form

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \hat{E}(X_{r+t-j:r+2t}), \quad r = 1, 2, \ldots, n-2t, \tag{45}$$

i.e.

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \quad r = 1, 2, \ldots, n-2t, \tag{46}$$

which is an unbiased estimate of the r-th TL-moment $\lambda_r^{(t)}$. Let us note that for each $j = 0, 1, \ldots, r-1$, the values $x_{i:n}$ in (46) are not equal to zero only for $r + t - j \leq i \leq n - t - j$, taking combination numbers into account. A simple adjustment of equation (46) provides an alternative linear form

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{i=r+t}^{n-t} \left[\frac{\sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j}}{\binom{n}{r+2t}} \right] \cdot x_{i:n}. \tag{47}$$

For $r = 1$, for example, we obtain for the first sample TL-moment

$$l_1^{(t)} = \sum_{i=t+1}^{n-t} w_{i:n}^{(t)} \cdot x_{i:n}, \tag{48}$$

with the weights

$$w_{i:n}^{(t)} = \frac{\binom{i-1}{t} \cdot \binom{n-i}{t}}{\binom{n}{2t+1}}. \tag{49}$$

The above results can be used for the estimation of TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ by simple ratios

$$t_3^{(t)} = \frac{l_3^{(t)}}{l_2^{(t)}}, \tag{50}$$

$$t_4^{(t)} = \frac{l_4^{(t)}}{l_2^{(t)}}. \tag{51}$$

We can choose $t = n\alpha$, representing the size of the adjustment from each end of the sample, where α is a certain ratio, where $0 \leq \alpha < 0{,}5$.

Table 3 contains the expressions for TL-moments and their ratios as well as those for parameter estimations of logistic and Cauchy probability distributions obtained employing the method of TL-moments ($t = 1$); for more, see, e.g.[15].

Table-3. Formulas for TL-moments and their ratios and parameter estimations made by the method of TL-moments of logistic and Cauchy probability distributions ($t = 1$)

Distribution	TL-moments and ratios of TL-moments	Parameter estimation
Logistic	$\lambda_1^{(1)} = \mu$ $\lambda_2^{(1)} = 0{,}500\,\sigma$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0{,}083$	$\hat{\mu} = l_1^{(1)}$ $\hat{\sigma} = 2\,l_2^{(1)}$
Cauchy	$\lambda_1^{(1)} = \mu$ $\lambda_2^{(1)} = 0{,}698\,\sigma$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0{,}343$	$\hat{\mu} = l_1^{(1)}$ $\hat{\sigma} = \dfrac{l_2^{(1)}}{0{,}698}$

Source: Kyselý and Picek [15] calculations

2.3. Maximum Likelihood Method

Let a random sample of sample size n come from the three-parametric lognormal distribution with a probability density function

$$f(x;\mu,\sigma^2,\theta) = \frac{1}{\sigma\cdot(x-\theta)\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{[\ln(x-\theta)-\mu]^2}{2\sigma^2}\right], \qquad x>\theta, \tag{52}$$
$$= 0,$$

where $-\infty < \mu < \infty$, $\sigma^2 > 0$, $-\infty < \theta < \infty$ are parameters. Three-parametric lognormal distribution is described in detail, for example, in Ulrych, et al. [4], Bílková [8], Bílková [9], Bílková [11] and Bílková [12].

We obtain the likelihood function

$$L(\boldsymbol{x};\mu,\sigma^2,\theta) = \prod_{i=1}^{n} f(x_i;\mu,\sigma^2,\theta) =$$

$$= \frac{1}{(\sigma^2)^{n/2}\cdot(2\pi)^{n/2}\cdot\prod_{i=1}^{n}(x_i-\theta)}\cdot\exp\left\{\sum_{i=1}^{n}-\frac{[\ln(x_i-\theta)-\mu]^2}{2\sigma^2}\right\}. \tag{53}$$

We determine the natural logarithm of the likelihood function

$$\ln L(\boldsymbol{x};\mu,\sigma^2,\theta) = \sum_{i=1}^{n}-\frac{[\ln(x_i-\theta)-\mu]^2}{2\sigma^2} - \frac{n}{2}\cdot\ln\sigma^2 - \frac{n}{2}\cdot\ln(2\pi) - \sum_{i=1}^{n}\ln(x_i-\theta). \tag{54}$$

We make the first partial derivatives of the likelihood function logarithm according to μ and σ^2 equal to zero, obtaining a system of likelihood equations

$$\frac{\partial\ln L(\boldsymbol{x};\mu,\sigma^2,\theta)}{\partial\mu} = \frac{\sum_{i=1}^{n}[\ln(x_i-\theta)-\mu]}{\sigma^2} = 0, \tag{55}$$

$$\frac{\partial \ln L(\boldsymbol{x};\mu,\sigma^2,\theta)}{\partial \sigma^2}=\frac{\sum_{i=1}^{n}[\ln(x_i-\theta)-\mu]^2}{2\sigma^4}-\frac{n}{2\sigma^2}=0. \tag{56}$$

Solving is maximum likelihood estimations of parameters μ and σ^2 for the parameter θ

$$\hat{\mu}(\theta)=\frac{\sum_{i=1}^{n}\ln(x_i-\theta)}{n}, \tag{57}$$

$$\hat{\sigma}^2(\theta)=\frac{\sum_{i=1}^{n}[\ln(x_i-\theta)-\hat{\mu}(\theta)]^2}{n}. \tag{58}$$

If the value of the parameter θ is known, we get maximum likelihood estimates of the remaining two parameters of the three-parametric lognormal distribution using equations (57) and (58). However, if the value of the parameter θ is unknown, the problem is more complicated. It has been proved that if the parameter θ gets closer to $\min\{X_1, X_2, \ldots, X_n\}$, then the likelihood function approaches infinity. The maximum likelihood method is also often combined with the Cohen method, where the smallest sample value is made equal to $100 \cdot (n+1)^{-1}$% quantile

$$x_{\min}^{V}=\hat{\theta}+\exp(\hat{\mu}+\hat{\sigma}\cdot u_{(n+1)^{-1}}). \tag{59}$$

Equation (59) is then combined with the system of equations (57) and (58). For the solution of maximum likelihood equations (57) and (58), it is also possible to use $\hat{\theta}$ satisfying the equation

$$\sum_{i=1}^{n}(x_i-\hat{\theta})+\frac{\sum_{i=1}^{n}\frac{z_i^{\backslash}}{(x_i-\hat{\theta})}}{\hat{\sigma}(\hat{\theta})}=0, \tag{60}$$

where

$$z_i^{\backslash}=\frac{\ln(x_i-\hat{\theta})-\hat{\mu}(\hat{\theta})}{\hat{\sigma}(\hat{\theta})}, \tag{61}$$

where $\hat{\mu}(\hat{\theta})$ and $\hat{\sigma}(\hat{\theta})$ comply with equations (57) and (58), the parameter θ being replaced by $\hat{\theta}$. We may also obtain the bounds of variances

$$n\cdot D(\hat{\theta})=\frac{\sigma^2\cdot\exp(2\mu)}{\omega\cdot\left[\omega\cdot(1+\sigma^2)-2\sigma^2-1\right]}, \tag{62}$$

$$n\cdot D(\hat{\mu})=\frac{\sigma^2\cdot[\omega\cdot(1+\sigma^2)-2\sigma^2]}{\omega\cdot(1+\sigma^2)-2\sigma^2-1}, \tag{63}$$

$$n\cdot D(\hat{\sigma})=\frac{\sigma^2\cdot[\omega\cdot(1+\sigma^2)-1]}{\omega\cdot(1+\sigma^2)-2\sigma^2-1}. \tag{64}$$

3. OUTCOMES

L-moments method used to be employed in hydrology, climatology and meteorology in the research of extreme precipitation, see, e.g. [15], having mostly used smaller data sets. This study presents applications of L-moments and TL-moments to large sets of economic data, Table 4 showing the sample sizes of obtained household sample sets. Researched sampled sets of households constitute a reprezentative sample of the study population. The research variable is the net annual household income per capita (in CZK) in the Czech Republic (nominal income). The data collected by the Czech Statistical Office come from the EU-SILC survey (The European Union Statistics on Income and Living Conditions) spanning the period 2004–2007. In total, 96 income distributions were analyzed – for all households in the Czech Republic as well as with the use of particular criteria: gender, region (Bohemia and Moravia), social group, municipality size, age and the highest educational attainment.

Table-4. Sample sizes of income distributions

	2004	**2005**	**2006**	**2007**
Sample size	4,351	7,483	9,675	11,294

Source: Own calculations

Value $\alpha = 0.25$ from the middle of the interval $0 \leq \alpha < 0{,}5$ was used in this research. With only minor exceptions, the TL-moments method produced the most accurate results. L-moments was the second most effective method in more than half of the cases, the differences between this method and that of maximum likelihood not being significant enough as far as the number of cases, when the former gave better results than the latter. Table 5 represents distinctive outcomes for all 96 income distributions, showing the results for the total household sets in the Czech Republic.

Table-5. Parameter estimations of three-parametric lognormal curves obtained using three various methods of point parameter estimation and the value of χ^2 criterion

Year	**Method of TL-moments**			**Method of L-moments**			**Maximum likelihood method**		
	μ	**σ²**	**θ**	**μ**	**σ²**	**θ**	**μ**	**σ²**	**θ**
2004	10.961	0.552	39,899	11.028	0.675	33,738	11.503	0.665	7.675
2005	11.006	0.521	40,956	11.040	0.677	36,606	11.542	0.446	-8.826
2006	11.074	0.508	44,941	11.112	0.440	40,327	11.623	0.435	-42.331
2007	11.156	0.472	48,529	11.163	0.654	45,634	11.703	0.421	-171.292
Year	Criterion χ^2			Criterion χ^2			Criterion χ^2		
2004	494.441			866.279			524.478		
2005	731.225			899.245			995.855		
2006	831.667			959.902			1,067.789		
2007	1,050.105			1,220.478			1,199.035		

Source: Own calculations

Apart from the estimated parameter values of the three-parametric lognormal distribution, which were obtained having simultaneously employed TL-moments, L-moments and maximum likelihood methods,

Table 5 contains the values of the test criterion χ^2, indicating that the L-moments method produced – in two out of four cases – more accurate results than the maximum likelihood method, the most accurate outcomes in all four cases being produced by the TL-moments method. Similarly, as in the case of moment method of parametric estimation, the parameters of three-parametric lognormal curves were estimated in the way that we have given the first three theoretical L-moments (TL-moments) to equal to the corresponding L-moments (TL-moments) of sample distribution.

4. CONCLUSION

A relatively new class of moment characteristics of probability distributions has been introduced in the present paper. They are the characteristics of the location (level), variability, skewness and kurtosis of probability distributions constructed with the use of L-moments and TL-moments that represent a robust extension of L-moments. The very L-moments were implemented as a more robust alternative to classical moments of probability distributions. L-moments and their estimates, however, are lacking in some robust features that are associated with TL-moments. The results show that TL-moment method is best as compared with the other method especially as compare to maximum likelihood method. Sample TL-moments are the linear combinations of sample order statistics assigning zero weight to a predetermined number of sample outliers. They are unbiased estimates of the corresponding TL-moments of probability distributions. Some theoretical and practical aspects of TL-moments are still the subject of both current and future research. The efficiency of TL-statistics depends on the choice of α, for example, $l_1^{(0)}, l_1^{(1)}, l_1^{(2)}$ have the smallest variance (the highest efficiency) among other estimates for random samples from the normal, logistic and double exponential distribution. The recommendations from this research consists in the high accuracy of the methods of L-moments and especially TL-moments compared to other methods of parametric estimations.

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REFERENCES

[1] J. R. M. Hosking, "L-moments: Analysis and estimation of distributions using linear combinations of order statistics," *Journal of the Royal Statistical Society (Series B)*, vol. 52, pp. 105–124, 1990.

[2] R. J. Serfling, *Approximation theorems of mathematical statistics*. New York: John Wiley & Sons, 1980.

[3] K. Adamowski, "Regional analysis of annual maximum and partial duration flood data by nonparametric and l-moment methods," *Journal of Hydrology*, vol. 229, pp. 219–231, 2000.

[4] T. J. Ulrych, D. R. Velis, A. D. Woodbury, and M. D. Sacchi, "L-moments and C-moments," *Stochastic Environmental Research and Risk Assessment*, vol. 14, pp. 50–68, 2000.

[5] D. Bílková, "Modelling of wage distributions using the method of L-moments," presented at the Paper Presented at AMSE – Applications of Mathematics and Statistics in Economy Held on 25–28 August 2010, Demänovská Dolina, 2010.

[6] D. Bílková, "Use of the L-moments method in modeling the wage distribution," presented at the Paper Presented at Aplimat Held on 01–04 February 2011, Bratislava, 2011a.

[7] D. Bílková, "L-moments and their use in modeling the distribution of income and wage," Paper Presented at ISI Held on 21–26 August 2011. Dublin, 2011b.

[8] D. Bílková, "Modeling of income and wage distribution using the method of L-moments of parameter estimation," presented at the Paper Presented at International Days of Statistics and Economics at VŠE Held on 22–23 September 2011, Prague, 2011c.

[9] D. Bílková, "Three-parametric lognormal distribution and estimating its parameters using the method of L-moments," presented at the Paper Presented at RELIK – Reprodukce Lidského Kapitálu Held on 05–06 December 2011, Prague: CD, 2011d.

[10] D. Bílková, "Estimating parameters of lognormal distribution using the method of L-moments," *Research Journal of Economics, Business and ICT*, vol. 4, pp. 4–9, 2011e.

[11] D. Bílková, "Modelling of wage and income distributions using the method of L-moments," *Journal of Mathematics and System Science*, vol. 2, pp. 13–19, 2012a.

[12] D. Bílková, "Lognormal distribution and using l-moment method for estimating its parameters," *International Journal of Mathematical Models and Methods in Applied Sciences*, vol. 6, pp. 30–44, 2012b.

[13] D. Bílková, "Lognormal distribution parameter estimating using L-moments," *Journal of Mathematics and Technology*, vol. 3, pp. 33–51, 2012c.

[14] D. Bílková and I. Malá, "Application of the L-moment method when modelling the income distribution in the czech republic," *Austrian Journal of Statistics*, vol. 41, pp. 125–132, 2012.

[15] J. Kyselý and J. Picek, "Regional growth curves and improved design value estimates of extreme precipitation events in the czech republic," *Climate Research*, vol. 33, p. 243–255, 2007.