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CONTIGUOUS FUNCTION RELATIONS AND AN INTEGRAL REPRESENTATION FOR APPELL k-SERIES $F_{1,k}$

Shahid Mubeen[1†] --- Sana Iqbal[2] --- Gauhar Rahman[3]
[1,2,3]*Department of Mathematics, University of Sargodha, Sargodha, Pakistan*

ABSTRACT
The main objective of this paper is to derive contiguous function relations or recurrence relations and obtain an integral representation Appell k-series $F_{1,k}$, where $k > 0$.

Keywords: Pochhammer k-symbol, k-gamma function, k-beta function, Contiguous functions, Appell k-series.

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Contribution/ Originality

This study originates a new formula for Appell's series in the form of a new symbol $k > 0$ and contributes for deriving contiguous function relations, obtaining an integral representation of the Appell's series in terms of said symbol $k > 0$.

1. INTRODUCTION

In this section, we present the following fundamental relations for Pochhammer k-symbol, k-gamma and k-beta functions introduced by the researchers [1-6].

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, k > 0, x \in C, \quad kZ^- \tag{1.1}$$

and also gave the properties of said functions. The Γ_k is one parameter deformation of the classical gamma function such that $\Gamma_k \to \Gamma$ as $k \to 1$. The Γ_k is based on the repeated appearance of the expression of the following form

$$\alpha(\alpha+k)(\alpha+2k)(\alpha+3k)...(\alpha+(n-1)k). \tag{1.2}$$

The function of the variable α given by the statement (1.2), denoted by $(\alpha)_{n,k}$ is called the Pochhammer k-symbol. We obtain the usual Pochhammer symbol $(\alpha)_n$ by taking $k = 1$. The definition given in relation (1.1), is the generalization of $\Gamma(x)$and the integral form of Γ_k is given by

† Corresponding author

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \ Re(x) > 0. \tag{1.3}$$

From relation (1.3), we can easily show that

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma(\frac{x}{k}). \tag{1.4}$$

The same authors defined the k-beta function as

$$\beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \ Re(x) > 0, \ Re(y) > 0 \tag{1.5}$$

and the integral form of $\beta_k(x, y)$ is

$$\beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt. \tag{1.6}$$

From the definition of $\beta_k(x, y)$ given in relations (1.5) and (1.6), we can easily prove that

$$\beta_k(x, y) = \frac{1}{k} \beta(\frac{x}{k}, \frac{y}{k}). \tag{1.7}$$

They also have worked on the generalized k -gamma and k beta functions and discussed the following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x) \tag{1.8}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} \tag{1.9}$$

$$\Gamma_k(k) = 1, \ k > 0 \tag{1.10}$$

$$\Gamma_k(x) = a^{\frac{x}{k}} \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}a} dt, \ a \in R \tag{1.11}$$

$$\Gamma_k(\alpha k) = k^{\alpha-1}\Gamma(\alpha), \ k > 0, \ \alpha \in R \tag{1.12}$$

$$\Gamma_k(nk) = k^{n-1}(n-1)!, \ k > 0, \ n \in N \tag{1.13}$$

$$\Gamma_k((2n+1)\frac{k}{2}) = k^{\frac{2n-1}{2}} \frac{(2n)!\sqrt{\pi}}{2^n n!}, \ k > 0, \ n \in N \tag{1.14}$$

Using the relations (1.5) and (1.7), we see that, for $x, y > 0$ and $k > 0$, the following properties of k-beta function are satisfied:

$$\beta_k(x+k, y) = \frac{x}{x+y} \beta_k(x, y) \tag{1.15}$$

$$\beta_k(x, y+k) = \frac{y}{x+y}\beta_k(x, y) \tag{1.16}$$

$$\beta_k(xk, yk) = \frac{1}{k}\beta(x, y) \tag{1.17}$$

$$\beta_k(mk, mk) = \frac{[(m-1)!]^2}{k(2m-1)!},\ m \in R. \tag{1.18}$$

$$\beta_k(x, k) = \frac{1}{x}, \quad \beta_k(k, y) = \frac{1}{y}. \tag{1.19}$$

Note that when $k \to 1$, $\beta_k(x,y) \to \beta(x,y)$.

For more details about the theory of k-special functions like, k-gamma function, k-beta function, k-hypergeometric functions, solutions of k-hypergeometric differential equations, contagious functions relations, inequalities with applications and integral representations with applications involving k-gamma and k-beta functions and so forth (See [7-14]).

Driver and Johnston [15] determined the integral representation of generalized hypergeometric functions ${}_{m+1}F_m$. Habibullah and Mubeen [16] gave an integral representation of extended confluent hypergeometric functions ${}_mF_m$. Very resently, Mubeen and Habibullah [17] also obtained an integral representations of generalized k-hypergeometric and extended conuent k-hypergeometric functions.

2. APPELL k-SERIES

By using the definition of Pochhammer k -symbol and k -gamma function, we define k-Appell series or bivariate k-hypergeometric function with three parametersa, b_1, b_2 in numerator one parameter c in denominator as

$$F_{1,k}(a, b_1, b_2; c; z_1, z_2) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\, z_1^m z_2^n}{(c)_{m+n,k}\, m!n!} \tag{2.1}$$

for all $a, b_1,\ b_2, c \in \mathbb{C}$, $c \neq 0, -1, -2, -3, \ldots$, $|z_1|, |z_2| < 1$ and $k > 0$.

If $k = 1$, then (2.1) reduces to usual Appell's series or bivariate hypergeometric series $F_1(a, b_1, b_2; c;\ z_1,\ z_2)$ as

$$F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_2)_m(b_2)_n\, z_1^m z_2^n}{(c)_{m+n}\, m!n!};$$

for all $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, -3, \ldots$, $|z_1|, |z_2| < 1$, where $(a)_n = (a)(a+1)(a+2)\ldots.(a+n-1)$ and $(a)_0 = 1$ (see Appell [18]).

2.1. Contiguous Relations for $F_{1,k}$.

Two hypergeometric functions are said to be contiguous if their parameters a, b and c differ by integers. The relations made by contiguous functions are said to be contiguous function relations.

Mubeen, et al. [8] determined the contiguous function relations for k-hypergeometric functions with one parameter corresponding to Gauss fifteen contiguous function relations for hypergeometric functions and also they obtained contiguous function relations for two parameters.

Now if we increase or decrease one or more parameter of k-Appel series

$$F_{1,k}(a,b_1,b_2;c;z_1,z_2)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(c)_{m+n,k}\,m!n!}$$

by $\pm k$ where $k > 0$, then the resultant function is said to be contiguous to $F_{1,k}$.

For simplicity we use the following notations:

$$F_{1,k}=F_{1,k}(a,b_1,b_2;c;z_1,z_2)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(c)_{m+n,k}\,m!n!} \quad (2.2)$$

$$F_{1,k}(a+)=F_{1,k}(a+k,b_1,b_2;c;z_1,z_2)=\sum_{m,n=0}^{\infty}\frac{(a+k)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(c)_{m+n,k}\,m!n!} \quad (2.3)$$

and

$$F_{1,k}(a-)=F_{1,k}(a-k,b_1,b_2;c;z_1,z_2)=\sum_{m,n=0}^{\infty}\frac{(a-k)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(c)_{m+n,k}\,m!n!} \quad (2.4)$$

Similarly we can write the notations for $F_{1,k}(b+)$; $F_{1,k}(b-)$; $F_{1,k}(c+)$and $F_{1,k}(c-)$.

Thus we have the following eight contiguous functions for the Appell series $F_{1,k}$, where $k > 0$.

$$\begin{cases}
F_{1,k}(a+)=\sum_{m,n=0}^{\infty}\frac{(a+mk+nk)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{a(c)_{m+n,k}\,m!n!},\\
F_{1,k}(a-)=\sum_{m,n=0}^{\infty}\frac{(a-k)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(a+(m+n-1)k)(c)_{m+n,k}\,m!n!},\\
F_{1,k}(b_1+)=\sum_{m,n=0}^{\infty}\frac{(b_1+mk)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{b_1(c)_{m+n,k}\,m!n!},\\
F_{1,k}(b_1-)=\sum_{m,n=0}^{\infty}\frac{(b_1-k)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(b_1+(m-1)k)(c)_{m+n,k}\,m!n!},\\
F_{1,k}(b_2+)=\sum_{m,n=0}^{\infty}\frac{(b_2+nk)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{b_2(c)_{m+n,k}\,m!n!},\\
F_{1,k}(b_2-)=\sum_{m,n=0}^{\infty}\frac{(b_2-k)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(b_2+(n-1)k)(c)_{m+n,k}\,m!n!},\\
F_{1,k}(c+)=\sum_{m,n=0}^{\infty}\frac{c(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(c+mk+nk)(c)_{m+n,k}\,m!n!},\\
F_{1,k}(c+)=\sum_{m,n=0}^{\infty}\frac{(c+(m+n-1)k)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(c-k)(c)_{m+n,k}\,m!n!}.
\end{cases} \quad (2.5)$$

Now with the help of differential $k\theta_1 = kz_1\frac{d}{dz_1}$ and $k\theta_2 = kz_2\frac{d}{dz_2}$, we derive the following results:

$$(k\theta_1+k\theta_2+a)F_{1,k}=(k\theta_1+k\theta_2+a)\sum_{m,n=0}^{\infty}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}\,z_1^m z_2^n}{(c)_{m+n,k}\,m!n!}$$

or

$$(k\theta_1 + k\theta_2 + a)F_{1,k} = \sum_{m,n=0}^{\infty} \frac{(a+mk+nk)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k} z_1^m z_2^n}{(c)_{m+n,k} m!n!}$$

Hence with the aid of (2.4), it follows that

$$(k\theta_1 + k\theta_2 + a)F_{1,k} = aF_{1,k}(a+) \tag{2.6}$$

similarly

$$(k\theta_1 + b_1)F_{1,k} = b_1 F_{1,k}(b_1+) \tag{2.7}$$

and

$$(k\theta_2 + b_2)F_{1,k} = b_2 F_{1,k}(b_2+) \tag{2.8}$$

3. CONTIGUOUS FUNCTION RELATIONS FOR APPELL k-SERIES $F_{1,k}$

In this section, we have to obtain the following four contiguous functions relations for k-Appell series $F_{1,k}$, where $k > 0$:

3.1. Relation

$$(a - b_1 - b_2)F_{1,k}(a, b_1, b_2, c; z_1, z_2) - aF_{1,k}(a+k, b_1, b_2, c; z_1, z_2)$$
$$+ b_1 F_{1,k}(a, b_1+k, b_2, c; z_1, z_2) + b_2 F_{1,k}(a, b_1, b_2+k, c; z_1, z_2) = 0$$

Proof. By subtracting equations (2.6) and (2.7) from equation (2.5), we get

$$(k\theta_1 + k\theta_2 + a - k\theta_1 - b_1 - k\theta_2 - b_2)F_{1,k} = aF_{1,k}(a+) - b_1 F_{1,k}(b_1+) - b_2 F_{1,k}(b_2+).$$

This implies that

$$(a - b_1 - b_2)F_{1,k} - aF_{1,k}(a+) + b_1 F_{1,k}(b_1+) + b_2 F_{1,k}(b_2+) = 0$$

or

$$(a - b_1 - b_2)F_{1,k}(a, b_1, b_2, c; z_1, z_2) - aF_{1,k}(a+k, b_1, b_2, c; z_1, z_2)$$
$$+ b_1 F_{1,k}(a, b_1+k, b_2, c; z_1, z_2) + b_2 F_{1,k}(a, b_1, b_2+k, c; z_1, z_2) = 0$$

3.2. Relation

$$cF_{1,k}(a, b_1, b_2, c; z_1, z_2) - (c-a)F_{1,k}(a, b_1, b_2, c+k; z_1, z_2)$$
$$- a_1 F_{1,k}(a+k, b_1, b_2, c+k; z_1, z_2) = 0$$

Proof. Consider

$$F_{1,k}(a+k,b_1,b_2;c+k;z_1,z_2)=\sum_{m,n=0}^{\infty}\frac{c(a+mk+nk)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{a(c+mk+nk)(c)_{m+n,k}m!n!}$$

$$=\frac{c}{a}\sum_{m,n=0}^{\infty}\frac{(a+mk+nk)}{(c+mk+nk)}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}m!n!} \tag{3.1}$$

Now since

$$\frac{a+mk+nk}{c+mk+nk}=1-\frac{c-a}{c+mk+nk}$$

using this result in equation (4.1), we obtain

$$F_{1,k}(a+k,b_1,b_2;c+k;z_1,z_2)=\frac{c}{a}\sum_{m,n=0}^{\infty}[1-\frac{c-a}{c+mk+nk}]\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}m!n!}$$

$$=\frac{c}{a}\sum_{m,n=0}^{\infty}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}m!n!}-\frac{c}{a}\sum_{m,n=0}^{\infty}\frac{c-a}{c+mk+nk}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}m!n!}$$

Thus we obtain the required relation

$$cF_{1,k}(a,b_1,b_2,c;z_1,z_2)-(c-a)F_{1,k}(a,b_1,b_2,c+k;z_1,z_2)$$

$$-a_1F_{1,k}(a+k,b_1,b_2,c+k;z_1,z_2)=0$$

or

$$cF_{1,k}-(c-a)F_{1,k}(c+)-a_1F_{1,k}(a+,c+)=0\,.$$

3.3. Relation

$$cF_{1,k}(a,b_1,b_2,c;z_1,z_2)+c(1-kz_1)F_{1,k}(a,b_1+k,b_2,c;z_1,z_2)$$

$$-(c-a)kz_1F_{1,k}(a,b_1+k,b_2,c+k;z_1,z_2)=0$$

Proof. By applying the differential operator $k\theta_1=kz_1\frac{d}{dz_1}$, we have

$$k\theta_1F_{1,k}(a,b_1,b_2;c;z_1,z_2)=\sum_{m,n=0}^{\infty}\frac{mk(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}m!n!}$$

$$= k \sum_{m\geq 1, n=0}^{\infty} \frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k} z_1^m z_2^n}{(c)_{m+n,k}(m-1)!n!}$$

shifting m with $m+1$, we get

$$k\theta_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2) = kz_1 \sum_{m,n=0}^{\infty} \frac{(a)_{m+n+1,k}(b_1)_{m+1,k}(b_2)_{m,k} z_1^m z_2^n}{(c)_{m+n+1,k}\, m!n!} \tag{3.2}$$

Now since

$$(a)_{m+n+1,k} = (a)_{m+n,k}(a+mk+nk),$$
$$(c)_{m+n+1,k} = (c)_{m+n,k}(c+mk+nk)$$

and

$(b_1)_{m+1,k} = (b_1)_{m,k}(b_1+mk),$

thus using these results in equation (4.2) , we obtain

$$k\theta_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2) = kz_1 \sum_{m,n=0}^{\infty} \frac{(a+mk+nk)(b_1+mk)(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k} z_1^m z_2^n}{(c+mk+nk)(c)_{m+n,k}\, m!n!} \tag{3.3}$$

since

$$\frac{(a+mk+nk)(b_1+mk)}{c+mk+nk} = mk + (a+b_1-c) + \frac{(c-a)(c-b_1+nk)}{c+mk+nk}$$

using this result in equation (4.3), we have

$$k\theta_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2) = kz_1 \sum_{m,n=0}^{\infty} \left[mk + (a+b_1-c) + \frac{(c-a)(c-b_1+nk)}{c+mk+nk} \right] \frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k} z_1^m z_2^n}{(c)_{m+n,k}\, m!n!},$$

$$= kz_1 \sum_{m,n=0}^{\infty} mk \frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k} z_1^m z_2^n}{(c)_{m+n,k}\, m!n!} + (a+b_1-c)kz_1 \sum_{m,n=0}^{\infty} \frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k} z_1^m z_2^n}{(c)_{m+n,k}\, m!n!}$$

$$+(c-a)kz_1 \sum_{m,n=0}^{\infty} \frac{(c-b_1+nk)}{c+mk+nk} \frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k} z_1^m z_2^n}{(c)_{m+n,k}\, m!n!},$$

this implies that

$$k\theta_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2) = k^2 z_1 \theta_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2) + (a+b_1-c)kz_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)$$

$$+(c-a)kz_1 \sum_{m,n=0}^{\infty} \frac{(c-b_1+nk)}{c+mk+nk} \frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k} z_1^m z_2^n}{(c)_{m+n,k}\, m!n!}. \tag{3.4}$$

Now using the result $\frac{c-b_1+nk}{c+mk+nk} = 1 - \frac{b_1+mk}{c+mk+nk}$ in equation (4.4), we obtain

$$k\theta_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)=k^2 z_1\theta_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)+(a+b_1-c)kz_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)$$

$$+(c-a)kz_1\sum_{m,n=0}^{\infty}\left[1-\frac{b_1+mk}{c+mk+nk}\right]\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}m!n!},$$

this implies that

$$k\theta_1(1-kz_1)F_{1,k}(a,b_1,b_2;c;z_1,z_2)=(a+b_1-c)kz_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)$$

$$+(c-a)kz_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)-(c-a)kz_1\sum_{m,n=0}^{\infty}\frac{b_1+mk}{c+mk+nk}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}m!n!},$$

$$=b_1 kz_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)-(c-a)kz_1\sum_{m,n=0}^{\infty}\frac{b_1+mk}{c+mk+nk}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}m!n!},$$

$$=b_1 kz_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)-(c-a)kz_1\frac{b_1}{c}F_{1,k}(a,b_1+k,b_2;c+k;z_1,z_2). \tag{3.5}$$

Now by using equation (2.6), we have

$$k(1-kz_1)\theta_1 F_{1,k}=-b_1(1-kz_1)F_{1,k}+b_1(1-kz_1)F_{1,k}(b_1+) \tag{3.6}$$

substituting equation (3.6) in equation (3.5), we get

$$-b_1(1-kz_1)F_{1,k}+b_1(1-kz_1)F_{1,k}(b_1+)=b_1 kz_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)-(c-a)kz_1\frac{b_1}{c}F_{1,k}(b_1+;c+),$$

$$c(1-kz_1)F_{1,k}(b_1+)=ckz_1 F_{1,k}(a,b_1,b_2;c;z_1,z_2)-(c-a)kz_1 F_{1,k}(b_1+;c+)$$

which gives the required relation.

3.4. Relation

$$cF_{1,k}(a,b_1,b_2,c;z_1,z_2)+c(1-kz_2)F_{1,k}(a,b_1,b_2+k,c;z_1,z_2)$$

$$-(c-a)kz_2 F_{1,k}(a,b_1,b_2+k,c+k;z_1,z_2)=0.$$

Proof. By applying the differential operator $k\theta_2 = kz_2\frac{d}{dz_2}$ and then follows the same procedure used in relation (4.3) then the required relation will be obtain.

4. INTEGRAL REPRESENTATION OF APPELL k-SERIES $F_{1,k}$

In this section, we define the integral representation of k-Appell series $F_{1,k}$, where $k > 0$.

4.1. Integral Representation for Appell k-Series F_1.

The Appell's series F_1 can also be written as a one dimensional Eular-type integral: if $Re(c) > Re(a) > 0, k > 0$ then for all finite z_1, z_2

$$F_1[a,b_1,b_2,c;z_1,z_2]=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b_1}(1-zt)^{-b_2}\,dt$$

Now we define the integral representation of k-Appell series $F_{1,k}$ where $k > 0$.

4.2. Theorem

If $Re(c) > Re(a) > 0, k > 0$ then for all finite z_1, z_2

$$F_{1,k}[a,b_1,b_2,c;z_1,z_2]=\frac{\Gamma_k(c)}{k\Gamma_k(a)\Gamma_k(c-a)}\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{c-a}{k}-1}(1-kz_1t)^{\frac{-b_1}{k}}(1-kz_2t)^{\frac{-b_2}{k}}\,dt\,.$$

Proof. First note that

$$\frac{(a)_{m+n,k}}{(c)_{m+n,k}}=\frac{\Gamma_k(c)\Gamma_k(a+(m+n)k)}{\Gamma_k(a)\Gamma_k(c+(m+n)k)}$$

$$=\frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(c-a)}B_k(a+(m+n)k,c-a)$$

$$=\frac{\Gamma_k(c)}{k\Gamma_k(a)\Gamma_k(c-a)}\int_0^1 t^{\frac{a}{k}+(m+n)-1}(1-t)^{\frac{c-a}{k}-1}\,dt \tag{4.1}$$

Now using (1.11), (1.9) and (5.1), we get

$$F_{1,k}[a,b_1,b_2,c;z_1,z_2]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n,k}(b_1)_{m,k}(b_2)_{m,k}z_1^m z_2^n}{(c)_{m+n,k}\,m!n!}$$

$$=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n,k}}{(c)_{m+n,k}}\times\sum_{m,n=0}^{\infty}\frac{(b_1)_{m,k}(b_2)_{n,k}z_1^m z_2^n}{m!n!}$$

$$=\frac{\Gamma_k(c)}{k\Gamma_k(a)\Gamma_k(c-a)}\int_0^1 t^{\frac{a}{k}+(m+n)-1}(1-t)^{\frac{c-a}{k}-1}\,dt\times\sum_{m,n=0}^{\infty}\frac{(b_1)_{m,k}(b_2)_{n,k}z_1^m z_2^n}{m!n!}$$

$$=\frac{\Gamma_k(c)}{k\Gamma_k(a)\Gamma_k(c-a)}\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{c-a}{k}-1}dt\times\sum_{m,n=0}^{\infty}\frac{(b_1)_{m,k}(b_2)_{n,k}t^m z_1^m t^n z_2^n}{m!n!}$$

$$=\frac{\Gamma_k(c)}{k\Gamma_k(a)\Gamma_k(c-a)}\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{c-a}{k}-1}dt\times\int_0^1(1-kz_1t^m)^{\frac{-b_1}{k}}(1-kz_2t^n)^{\frac{-b_2}{k}}dt$$

$$=\frac{\Gamma_k(c)}{k\Gamma_k(a)\Gamma_k(c-a)}\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{c-a}{k}-1}(1-kz_1t^m)^{\frac{-b_1}{k}}(1-kz_2t^n)^{\frac{-b_2}{k}}dt\,.$$

Thus for $|z_1|,\ |z_2| < 1$, the Integral representation for Appell's series $F_{1,k}$ is given by

$$F_{1,k}[a,b_1,b_2,c;z_1,z_2]=\frac{\Gamma_k(c)}{k\Gamma_k(a)\Gamma_k(c-a)}\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{c-a}{k}-1}(1-kz_1t)^{\frac{-b_1}{k}}(1-kz_2t)^{\frac{-b_2}{k}}dt\,.$$

Remarks: In this paper if we letting $k \to 1$ then we obtain the contiguous or recurrence relations and an integral representation of Appell's series F_1.

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