International Journal of Mathematical Research
2015 Vol.4, No.2, pp.64-75
ISSN(e): 2306-2223
ISSN(p): 2311-7427
DOI: 10.18488/journal.24/2015.4.2/24.2.64.75

A GENRALIST PREDATOR PREY MATHEMATICAL MODEL ANALYSIS ON A CUSP POINT AND BOGDANOV-TAKENS BIFURCATION POINT

Temesgen Tibebu Mekonnen[1]
[1]*Department of Mathematics Debre Berhan University Ethiopia, Ethiopia*

ABSTRACT

This study considers a generalist predator-prey system. We investigate a local bifurcation namely Bogdanov-Takens (co dimension 2) bifurcations and a cusp point on two significant points in the division of parameter space. These points show the place where Bogdanov-Takens point and a cusp point exist. The existence of these bifurcations proved analytically by Normal form derivation. To reach the analysis we first studied the steady state solutions and their dependence on parameters and then investigate a parameter space which is divided into subregions based on the number of equilibrium points. We identified three vital parameters α stands for the maximum uptake rate of the generalist predator; θ stands for half saturation value and η such that μ/η is the conversion efficiency of the generalist predator where μ is the intrinsic growth rate of the predator.

Keywords: Couple differential equations, Parameter space, Cusp point, Bogdanov-takens bifurcation point, Normal form derivation.

Received: 13 August 2014/ **Revised:** 20 January 2015/ **Accepted:** 27 January 2015/ **Published:** 2 February 2015

Contribution/ Originality

This study contributes new findings in the field of a mathematical model. We investigated two bifurcation points namely Bogdanov-Takens bifurcation point and the cusp point on the generalist predator prey mathematical model by observing and analyzing the change of the behavior of the solutions of the couple of the differential equations if small change appears on the parameters of the considered model by using the normal form of derivation method.

1. INTRODUCTION

A bifurcation is a qualitative change in the behaviour of solutions as one or more parameters are varied. The parametric values at which these changes occur are called bifurcation points. If the qualitative change occurs in a neighbourhood of a fixed point or periodic solution, it is called a local bifurcation. Any other qualitative change that occurs is considered as a global bifurcation [1]. Here we can see two bifurcations namely Trans-critical bifurcation and Tangent bifurcation. Trans-critical bifurcation occurs in the mathematical model or ordinary differential equation $x' = \alpha x - x^2$ depends only on one parameter α and has two equilibrium points $x = 0$ *and* $x = \alpha$. For $\alpha < 0$ the equilibrium point $x = 0$ is stable where as $x = \alpha$ is unstable and for $\alpha > 0$ the equilibrium point $x = 0$ is unstable where as $x = \alpha$ is stable. This shows that the Trans-critical bifurcation experiences change of stability. The Tangent bifurcation corresponding to the creation and destruction of fixed points in one dimensional system given by a mathematical model $x' = x^2 - \alpha$ and has two equilibrium points $x = \pm\sqrt{\alpha}$. This shows that for $\alpha < 0$ there is no equilibrium point and for $\alpha > 0$ two equilibrium points created and the point $x = 0$ is the bifurcation point.

In this paper we investigate the local bifurcation involving the system dynamics. We find that the considered system is very rich in dynamics and involves several interesting bifurcations. A comprehensive analytical bifurcation analysis presented on the local bifurcation. The local bifurcation we studied thoroughly in this paper is Bogdanov-Takens (co dimension 2) bifurcation. The work done by Temesgen Tibebu Mekonnen [2] the author of this paper has investigated the existence of saddle node, Trans-critical and pitchfork bifurcations in the parameter space.

The division of the parameter space as shown in figures 3, 4 and 5 with respective parameter values $\eta = 1$, $\eta > 1$ and $\eta < 1$ are the fundamental bifurcation diagrams of this study.

In the next section we present the basic concepts: the model, steady state solutions and division of parameter space in the system dynamics. In section three, the central part of this study: the Bogdanov-Takens bifurcation is analysed. In section four Bogdanov-Takens (double zero eigen value) once again revisited and conclusions of the work presented in section five.

2. THE MODEL, STEADY STATE SOLUTIONS AND DIVISION OF PARAMETER SPACE

2.1. The Model

Let X and Y represent the density of Prey and Generalist Predator respectively with the assumption that each species grow logistically in the absence of the other. Further we assume that the predator's functional response is of Holling type II and hence the dynamics of the considered system is given by:

$$\frac{dX}{dt} = aX\left(1 - \frac{X}{K_1}\right) - \frac{bXY}{c+X} \qquad (1)$$

$$\frac{dY}{dt} = dY\left(1 - \frac{Y}{K_2}\right) + \frac{eXY}{c+X} \qquad (2)$$

The constants a (d), k_1 (k_2) are the intrinsic growth rate and carrying capacity of prey (predator). b, c stands for maximum uptake rate and half saturation value of the predator and $e = \delta b$ ($0 < \delta < 1$) where δ is the conversion efficiency. From the above model we clearly observe that the predator can survive in the absence of the prey and the per capita growth rate of the predator is enhanced by $\frac{eX}{c+X}$in the presence of prey. To reduce the number of parameters we non-dimensionalize the considered model (1,2) and obtained the following:

$$\frac{dx}{dt} = f(x)(g(x) - y) \tag{3}$$

$$\frac{dy}{dt} = \mu y(h(x) - y) \tag{4}$$

Where

$$f(x) = \frac{\alpha x}{\theta + x}) ; \; g(x) = \frac{(1-x)(\theta+x)}{\alpha}, h(x) = 1 + \frac{\alpha x}{\eta(\theta + x)} \tag{5}$$

With $= \frac{X}{k_1}, y = \frac{Y}{k_1}, t = \alpha T, \alpha = \frac{bk_2}{ak_1}, \mu = \frac{d}{a}, \eta = \frac{\mu k_2}{\delta k_1}, \theta = \frac{c}{k_1}$.

2.2. Nature of Steady State Solutions

In this section we study the existence of equilibrium solutions of (3, 4) and study their nature through linear analysis. Clearly the system admits (0, 0) as trivial equilibrium and (1, 0), (0, 1) to be its axial equilibrium points. The interior equilibrium points are the intersection points of the isoclines $\mathbf{y} = \mathbf{g}(\mathbf{x})$ and $\mathbf{y} = \mathbf{h}(\mathbf{x})$ in the interior of the positive quadrant. Following the standard linear analysis it is easy to observe that (0, 0) and (1, 0) are unstable node and saddle point respectively. We observe that the nature of (0, 1) depends on the values of the parameters α and θ. If $\theta / \alpha \leq 1$ then (0, 1) is stable and it becomes a saddle if $\theta / \alpha > 1$. In the latter case (0, 1) is unstable in the x-direction and stable in the y-direction. Thus we have the equilibria (0, 0), (1, 0) to be always hyperbolic. Whereas (0, 1) is hyperbolic when $\alpha \neq \theta$ and it turns to non hyperbolic when α equals θ. Analysing the Jacobian of the system (3, 4) at its interior equilibrium point $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ gives the associated characteristic equation to be

$$\boldsymbol{\lambda}^2 + [\mu \boldsymbol{y}^* - \boldsymbol{f}(\boldsymbol{x}^*)\boldsymbol{g}'(\boldsymbol{x}^*)]\boldsymbol{\lambda} + \mu \boldsymbol{y}^* \boldsymbol{f}(\boldsymbol{x}^*)[\boldsymbol{h}'(\boldsymbol{x}^*) - \boldsymbol{g}'(\boldsymbol{x}^*)] = \mathbf{0}$$

To understand the nature of an interior equilibrium solution of the system (3, 4) we need to study the signs of the trace $\mu \boldsymbol{y}^* - \boldsymbol{f}(\boldsymbol{x}^*)\boldsymbol{g}'(\boldsymbol{x}^*)$ and the determinant $\mu \boldsymbol{y}^* \boldsymbol{f}(\boldsymbol{x}^*)[\boldsymbol{h}'(\boldsymbol{x}^*) - \boldsymbol{g}'(\boldsymbol{x}^*)]$ which are respectively the sum and product of the eigen values of the considered Jacobian matrix [3].

2.3. Division of Parameter Space

From the qualitative behaviour of the isoclines of the system (3, 4) we can observe that there is a possibility for the system to admit multiple interior equilibrium solutions. The number of interior equilibrium solutions admitted by the considered system (3, 4) and its dependence on the involved parameters can be best understood by analysing the following cubic equation:

$$\boldsymbol{P(x) = x^3 + (2\theta - 1)x^2 + \left[\theta(\theta - 2) + \alpha\left(1 + \frac{\alpha}{\eta}\right)\right]x + \theta(\alpha - \theta) = 0} \quad (6)$$

This is obtained by equating the functions g(x) and h(x) (5). If x is a positive root of (5) then either (x, g(x)) or (x, h(x)) gives an interior equilibrium point of the system (3, 4). Thus the number of positive roots of (5) corresponds to the number of interior equilibrium solutions admitted by the system (3, 4). Hence from this cubic polynomial equation we observe that the system (3, 4) admits a maximum of three interior equilibria in the first quadrant of the phase space. The discriminant of the cubic polynomial equation (6) is

$$\Delta P = \theta^2(\alpha - \theta)^2 - \frac{2}{3}\theta(\alpha - \theta)(2\theta - 1)\left[\theta(\theta - 2) + \alpha\left(1 + \frac{\alpha}{\theta}\right)\right] + \frac{4}{27}\theta(\alpha - \theta)(2\theta - 1)^3 -$$

$$\frac{1}{27}(2\theta - 1)^2[\theta(\theta - 2) + \alpha\left(1 + \frac{\alpha}{\theta}\right)]^2 + \frac{4}{27}[\theta(\theta - 2) + \alpha\left(1 + \frac{\alpha}{\theta}\right)]^3. \quad (7)$$

The sign of ΔP determines the number of real roots admitted by (6). If ΔP is either < 0, = 0 or > 0 then $P(x) = 0$ admits three distinct real roots, three real roots with one of them repeated twice or three roots with one of them real respectively [4]. For a chosen set of parameters α, θ and η sign of the discriminant along with the signs of the coefficients of the equation $P(x)$ yield further information on the roots of the equation (6). For the sake of simplicity let us denote the constant term in (6) by

$$C_0(\theta, \alpha) = \theta(\theta - \alpha) \quad (8)$$

This represents the product of the roots of (6). Analyzing the curves $\Delta P = 0$ and $C_0(\theta, \alpha) = 0$ it can be verified that $C_0(\theta, \alpha) = 0$ is tangential to $\Delta P = 0$ at $\left(\frac{1}{(1+\frac{1}{\eta})}, \frac{1}{(1+\frac{1}{\eta})}\right)$ which is denoted by *c* and these two curves intersect at $(\sqrt{\eta}/2, \sqrt{\eta}/2)$ denoted by *e* (figures 2 and 3) which is a candidate of Bogdanov-Takens bifurcation point. From these figures we observe that there is another significant point $\left(\frac{1-2\theta}{3}, 1 + \frac{\alpha(1-2\theta)}{\eta(\theta+1)}\right)$ lying on the curve $\Delta P = 0$denoted by *f* at which the curve takes a sharp turn (cusp point). It is further observed that this point *f* also satisfies the equations

$$K1 = (2\theta - 1)\left[\theta(\theta - 2) + \alpha\left(1 + \frac{\alpha}{\theta}\right)\right] + 9\theta(\theta - \alpha) = 0 \quad (9)$$

$$K2 = (2\theta - 1)^2 + 3\left[\theta(\theta - 2) + \alpha\left(1 + \frac{\alpha}{\theta}\right)\right] = 0 \quad (10)$$

Implying that the equation (6) admits a triple root at *f* [4]. It is interesting to note that the points *c* and *e* merge in *f* when$\eta = 1$. It can be observed that, depending on the value of the parameter$(\eta = 1, \eta > 1\ or \eta < 1)$, the equations $\Delta P = 0$ and $C_0(\theta, \alpha) = 0$ divide the positive quadrant of the (θ, α) space into several significant regions as given below.

If $\boldsymbol{\eta = 1}$ the positive quadrant of the $(\boldsymbol{\theta, \alpha})$ space is divided into four regions (Figure 1) given by

Region I = $\{(\boldsymbol{\theta, \alpha}) / \ \boldsymbol{\alpha \geq \theta \ and \ \Delta P \leq 0}\}$

Region II = $\{(\boldsymbol{\theta, \alpha}) / \ \boldsymbol{\alpha \leq \theta \ and \ \Delta P \leq 0}\}$

Region III = $\{(\boldsymbol{\theta, \alpha}) \ / \ \boldsymbol{\alpha < \theta \ and \ \Delta P > 0}\}$

Region IV = $\{(\boldsymbol{\theta, \alpha}) / \ \boldsymbol{\alpha > \theta \ and \ \Delta P > 0}\}$

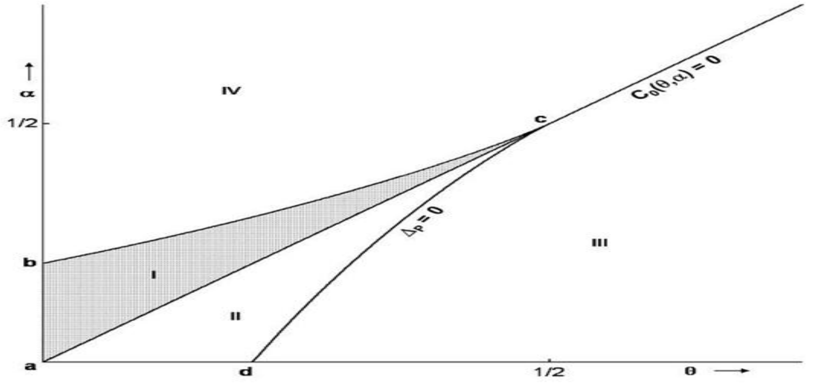

Figure-1. This figure represents division of the parameter space for $\eta = 1$. The regions bounded by $\boldsymbol{c_0(\theta, \alpha) = 0}$ and $\boldsymbol{\Delta P = 0}$ enclosed by {a,b,c} and {a,c,d} represent regions I and II respectively. Regions lying below (above) the line $\boldsymbol{c_0(\theta, \alpha) = 0}$ is Region III(IV). The system admits two interior equilibrium points in the region I, one interior equilibrium point in Regions II & III and no interior equilibrium point in region IV.

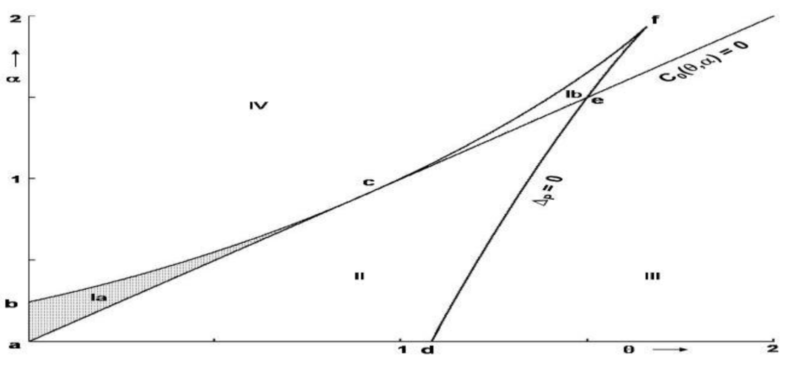

Figure-2. This figure represents division of the parameter space for $\eta > 1$. The regions bounded by $\boldsymbol{c_0(\theta, \alpha) = 0}$ and $\boldsymbol{\Delta P = 0}$ enclosed by {a,b,c}, {c,f,e} and {a,d,e} represents regions Ia, Ib and II respectively. Region lying below (above) the line $\boldsymbol{c_0(\theta, \alpha) = 0}$ is region III(Iv) for $\boldsymbol{\eta = 9}$ (a representative for $\eta > 1$). The system admits two interior equilibrium points in the region Ia, one interior equilibrium point in regions II & III and no interior equilibrium point in regions Ib& IV.

If $\eta > 1$ then region I is further divided into

Region Ia = $\{(\theta, \alpha) : \ \theta \leq \frac{1}{1+\frac{1}{\eta}},, \alpha \geq \theta, \Delta P \leq 0 \}$

Region Ib = $\{(\theta,\alpha) : \theta > \frac{1}{1+\frac{1}{\eta}},, \alpha \geq \theta, \Delta P \leq 0\}$

And regions II, III and IV remain as in the case $\eta = 1$. (Figure 2)
On the other hand if $0 < \eta < 1$ then region II is divided into

Region IIa = $\{(\theta,\alpha) : \theta \leq \frac{1}{1+\frac{1}{\eta}}, \alpha \leq \theta, \Delta P \leq 0\}$

Region IIb = $\{(\theta,\alpha) : \theta > \frac{1}{1+\frac{1}{\eta}},, \alpha \leq \theta, \Delta P \leq 0\}$

And the regions II, III and IV remain as in the case η = 1. (Figure 3).

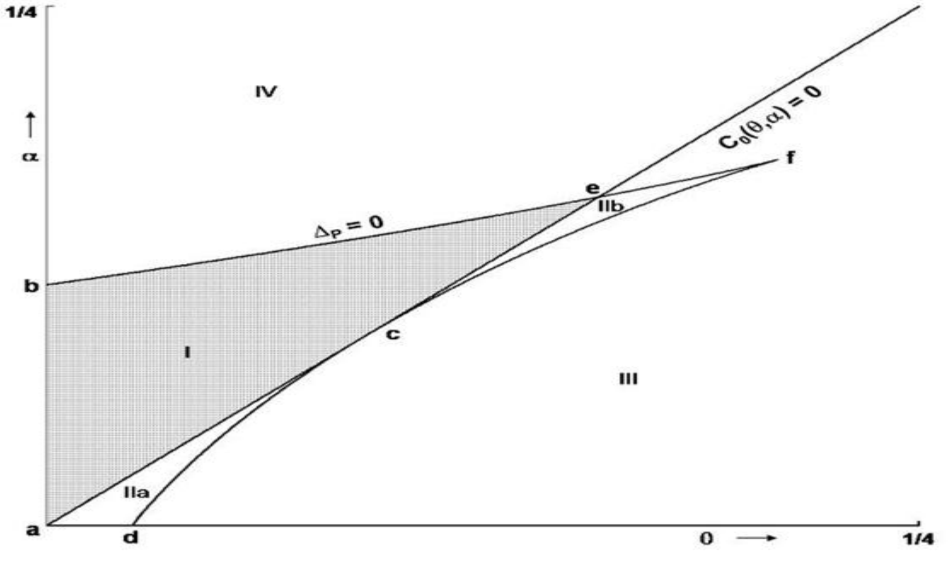

Figure-3. This figure represents division of the parameter space for 0<η< 1. The regions bounded by $\boldsymbol{c_0(\theta,\alpha) = 0}$ and $\boldsymbol{\Delta P = 0}$ enclosed by {a,b,e}, {a,c,d} and {c,e,f} represents regions I, IIa and IIb respectively. Region lying below (above) the line $\boldsymbol{c_0(\theta,\alpha) = 0}$ is region III(Iv) for $\boldsymbol{\eta = 0.1}$(a representative for 0 <η< 1) The system admits three interior equilibrium points in the region IIb, two interior equilibrium points in region I and no interior equilibrium point in regions IV.

From the nature of the boundary equilibrium solutions we observe that (0, 1) changes its stability nature as the parameter cross the curve $c_0(\theta,\alpha) = 0$indicating occurrence of bifurcation along the curve $C_0(\theta,\alpha) = 0$. Similarly we also observe change in the number of equilibrium solutions of the system as the parameter cross the discriminant curve $\Delta P = 0$presentinganother incidence of bifurcation. This curve also contains three significant points*c, e* and *f*. In the next section we study the significance of these curves and the importance of the points mentioned above.

2. BOGDANOV-TAKENS (DOUBLE ZERO EIGEN VALUE) BIFURCATION

Division of the parameter space as shown in figures 1, 2 and 3 enables us to recognize some basic bifurcations associated with the system as the parameters move from one region to another. The considered system experiences Saddle-Node bifurcation along the curve $\boldsymbol{\Delta P = 0}$, Trans-Critical bifurcation along the curve $\boldsymbol{c_0(\theta, \alpha) = 0}$ and Pitchfork bifurcation at the point *c* (only for $\boldsymbol{\eta} < 1$ *and* $\eta > 1$). In this work we will discuss only bifurcations experience at the points*e* which is a Bogdanov-Takens bifurcation point and *f* which is the cusp point. Now we show that Bogdanov-Takens bifurcation takes place at the points *e* and *f* (figures 1-3). This is achieved by applying a series of transformations on the considered model and using normal form theory [5]. Let $\boldsymbol{(x_1, y_1)}$ be an interior equilibrium point of the system (3, 4). We know that the Bogdanov-Takens bifurcation takes place at the equilibrium point $\boldsymbol{(x_1, y_1)}$ if the associated Jacobian matrix $\boldsymbol{J(x_1, y_1)}$ admits a double zero eigen value [5, 6]. Hence to ensure the occurrence of Bogdanov-Takens bifurcation at (x_1, y_1) we assume that

$$DetJ(x_1, y_1) = -\frac{\mu x_1 y_1 (1-\theta-2x_1)}{\theta + x_1} + \frac{\mu \alpha^2 \theta x_1 y_1}{\eta(\theta + x_1)^3} = 0 \tag{11}$$

$$TrJ(x_1, y_1) = -\mu y_1 + \frac{x_1(1-\theta-2x_1)}{\theta + x_1} \tag{12}$$

From (11) we have

$$1 - \theta - 2x_1 = \frac{\alpha^2 \theta}{\eta(\theta + x_1)^2} \tag{13}$$

Using (13) in (12) we obtain

$$\boldsymbol{\mu y_1 = \frac{\alpha^2 \theta x_1}{\eta(\theta + x_1)^3}} \tag{14}$$

Hence forth we shall assume that the equilibrium point $\boldsymbol{(x_1, y_1)}$ satisfies (13) and (14). Under this assumption we proceed to find the normal form for the system (3, 4) which requires application of a series of transformations. First we translate the interior equilibrium point $\boldsymbol{(x_1, y_1)}$ to the origin using the transformations: $\boldsymbol{u_1 = x - x_1}$, $\boldsymbol{u_2 = x - x_2}$ and $\boldsymbol{t = \eta(\theta + x_1)T}$ and hence obtain

$$\boldsymbol{u_1' = \frac{\alpha^2 \theta}{\eta(\theta + x_1)^2} u_1 - \alpha \eta x_1 u_2 + \eta(1 - \theta - 3x_1) u_1^2 - \alpha \eta u_1 u_2 + O(|u_1, u_2|)^3} \tag{15}$$

$$\boldsymbol{u_2' = \frac{\alpha^3 \theta^2 x_1}{\eta(\theta + x_1)^4} u_1 - \frac{\alpha^2 \theta x_1}{(\theta + x_1)^2} u_2 + \left(\frac{\alpha(\theta - x_1)}{\eta(\theta + x_1)} - 1\right) u_1 u_2 - \mu \eta(\theta + x_1) + O(|u_1, u_2|)^3} \tag{16}$$

The system (15, 16) can be conveniently represented as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} P(u_1, u_2) \\ Q(u_1, u_2) \end{pmatrix} + O(|u_1, u_2|^3) \tag{17}$$

Where

$$A = \begin{pmatrix} \dfrac{\alpha^2 \theta x_1}{(\theta + x_1)^2} & -\alpha \eta x_1 \\ \dfrac{\alpha^3 \theta^2 x_1}{\eta(\theta + x_1)^3} & \dfrac{\alpha^2 \theta x_1}{(\theta + x_1)^2} \end{pmatrix}$$

and $P(u_1, u_2)$, $Q(u_1, u_2)$ are terms of order 2. Observe that (0, 0) is an equilibrium point of the above system with the associated Jacobian being A. clearly A has double zero eigen value.

The second transformation is a linear transformation which transforms the vector $\boldsymbol{u}$ to $\boldsymbol{v}$through the following relation

$$\boldsymbol{v} = B\boldsymbol{u} \tag{18}$$

Where

$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$and B is a matrix of the form

$B = \begin{pmatrix} 1 & 0 \\ a_{21} & a_{22} \end{pmatrix}$where$a_{22} \neq 0$ and satisfies

$BAB^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Computing for B we obtain

$$B = \begin{pmatrix} 1 & 0 \\ \frac{\alpha^2\theta x_1}{(\theta + x_1)^2} & -\alpha\eta x_1 \end{pmatrix} \tag{19}$$

Clearly we have $a_{22} \neq 0$. In view of the transformation (18) we have

$$v_1 = u_1 \tag{20}$$

$$v_2 = \frac{\alpha^2\theta x_1}{(\theta + x_1)^2} u_1 - \alpha\eta x_1 u_2 \tag{21}$$

Differentiating (20) and (21) with respect to t we obtain

$$v_1' = u_2' \tag{22}$$

$$u_2' = \frac{\alpha^2\theta x_1}{(\theta + x_1)^2} u_1' - \alpha\eta x_1 u_2' \tag{23}$$

Substituting (15) and (16) into the system (22 , 23) we obtain

$$v_1' = u_2 + [\eta(1 - \theta - 3x_1) - \frac{\alpha^2\theta}{(\theta + x_1)^2}]u_1^2 + \frac{1}{x_1}u_1u_2 + O(|u|)^3$$

$$v_2' = [\frac{\alpha^2\theta\eta x_1}{(\theta + x_1)^2}(1 - \theta - 3x_1) - \frac{\alpha^4\theta^2 x_1}{(\theta + x_1)^4} - \left(\frac{\alpha(\theta - x_1)}{\eta(\theta + x_1)} - 1\right)\frac{\alpha^2\theta x_1}{(\theta + x_1)^2} + \frac{\mu\alpha^3\theta^2 x_1}{(\theta + x_1)^3}]u_1^2$$
$$+ \left[\frac{\alpha^2\theta}{(\theta + x_1)^2} + \frac{\alpha(\theta - x_1)}{\eta(\theta + x_1)} - \frac{2\mu\alpha\theta}{\theta + x_1} - 1\right]u_1u_2 + \frac{\mu(\theta + x_1)}{\alpha x_1^2}u_2^2 + O(|u|)^3$$

In view of (20) and (21) the above system can be written as

$$v_1' = \alpha_{11}v_2 + \alpha_{12}v_1^2 + \alpha_{13}v_1v_2 + O(|v|)^3 \tag{24}$$

$$v_2' = \alpha_{21}v_1^2 + \alpha_{22}v_1u_2 + \alpha_{23}v_2^2 + O(|v|)^3 \tag{25}$$

Where

$\alpha_{11} = 1,$

$$\alpha_{12} = \eta(1 - \theta - 3x_1) - \frac{\alpha^2\theta}{(\theta + x_1)^2}$$

$$\alpha_{13} = \frac{1}{x_1}$$

$$\alpha_{21} = \frac{\alpha^2\theta\eta x_1}{(\theta + x_1)^2}(1 - \theta - 3x_1) - \frac{\alpha^4\theta^2 x_1}{(\theta + x_1)^4} - \left(\frac{\alpha(\theta - x_1)}{\eta(\theta + x_1)} - 1\right)\frac{\alpha^2\theta x_1}{(\theta + x_1)^2} + \frac{\mu\alpha^3\theta^2 x_1}{(\theta + x_1)^3}$$

$$\alpha_{22} = \frac{\alpha^2\theta}{(\theta + x_1)^2} + \frac{\alpha(\theta - x_1)}{\eta(\theta + x_1)} - \frac{2\mu\alpha\theta}{\theta + x_1} - 1$$

$$\alpha_{23} = \frac{\mu(\theta + x_1)}{\alpha x_1^2}$$

Observe that the above system (24, 25) can also represented in the form

$$\boldsymbol{v} = Jv + F(v) \tag{26}$$

Where $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F(v) = \begin{pmatrix} \alpha_{12}u_1^2 + \alpha_{13}u_1u_2 \\ \alpha_{21}u_1^2 + \alpha_{22}u_1u_2 + \alpha_{23}u_2^2 \end{pmatrix} + O(|\boldsymbol{v}|)^3$

Now we employ the normal form theory to transform the system (26) to the following form

$$v_1' = v_2 + O(|\boldsymbol{v}|)^3 \tag{27}$$

$$v_2' = av_1^2 + bv_1v_2 + O(|\boldsymbol{v}|)^3 \tag{28}$$

With $ab \neq 0$

Which is nothing but the normal form associated with Bogdanov-Takens bifurcation [5]. The equation (26) can be written as

$$v' = Jv + F_2(v) + O(|\boldsymbol{v}|)^3 \tag{29}$$

Where

$$F_2(v) = \begin{pmatrix} \alpha_{12}v_1^2 + \alpha_{13}v_1v_2 \\ \alpha_{21}v_1^2 + \alpha_{22}v_1v_2 + \alpha_{23}v_2^2 \end{pmatrix} \tag{30}$$

Which consists of second degree terms. Consider the transformation

$$v = y + h_2(y) \tag{31}$$

Where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and $h_2(y) = O(|y|^2)$ is the space H_2 of all second degree polynomials of the form

$$h_2(y) = \begin{pmatrix} ay_1^2 + by_1y_2 + cy_2^2 \\ dy_1^2 + ey_1y_2 + fy_2^2 \end{pmatrix}$$

Differentiating (31) with respect to t we obtain

$$v' = y' + Dh_2(y)y' = (I + Dh_2(y))y' \tag{32}$$

where$I$ is the identity matrix. Substituting (31) and (32) in (29) we obtain

$$\big(I + Dh_2(y)\big)y' = J(y) + J\big(h_2(y)\big) + F_2(y) + O(|y|^3) \tag{33}$$

In a neighborhood of zero (33) can be written as

$$y' = \big(I + Dh_2(y)\big)^{-1}[J(y) + J\big(h_2(y)\big) + F_2(y) + O(|y|^3)]$$

as$\big(I + Dh_2(y)\big)^{-1}$ exists as $|y| \to 0$ and it is given by

$$\big(I + Dh_2(y)\big)^{-1} = I - Dh_2(y)Jy + F_2(y) + O(|y|^3) \tag{34}$$

Using (34) we have

$$y' = Jy + Jh_2(y) - Dh_2(y)Jy + F_2(y) + O(|y|^3) \tag{35}$$

Denoting

$$\overline{F_2}(y) = Jh_2(y) - Dh_2(y)Jy + F_2(y) \tag{36}$$

(35) reduces to the form

$$y' = Jy + \overline{F_2}(y) + O(|y|^3) \tag{37}$$

Now let us consider the linear transformation

$$L_J\big(h_2(y)\big) = Jh_2(y) - Dh_2(y)Jy \tag{38}$$

We have

$$L_J\big(h_2(y)\big) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} ay_1^2 + by_1y_2 + cy_2^2 \\ dy_1^2 + ey_1y_2 + fy_2^2 \end{pmatrix} - \begin{pmatrix} 2ay_1 + by_2 & by_1 + 2cy_2 \\ 2dy_1 + ey_2 & ey_1 + 2fy_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} dy_1^2 + (e-2a)y_1y_2 + (f-b)y_2^2 \\ -2dy_1y_2 - ey_2^2 \end{pmatrix} \tag{39}$$

In view of (30) and (39) equation (36) takes the form

$$\overline{F_2}(y) = Jh_2(y) - Dh_2(y)Jy + F_2(y)$$

$$= L_J\big(h_2(y)\big) + F_2(y)$$

$$= \begin{pmatrix} dy_1^2 + (e-2a)y_1y_2 + (f-b)y_2^2 \\ -2dy_1y_2 - ey_2^2 \end{pmatrix} + \begin{pmatrix} \alpha_{12}y_1^2 + \alpha_{13}y_1y_2 \\ \alpha_{21}y_1^2 + \alpha_{22}y_1y_2 + \alpha_{23}y_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} dy_1^2 + (e-2a)y_1y_2 + (f-b)y_2^2 + \alpha_{12}y_1^2 + \alpha_{13}y_1y_2 \\ -2dy_1y_2 - ey_2^2 + \alpha_{21}y_1^2 + \alpha_{22}y_1y_2 + \alpha_{23}y_2^2 \end{pmatrix} \tag{40}$$

By choosing the coefficients

$a = \frac{\alpha_{13}}{2}, d = -\alpha_{12}$ and $e = f - b = \alpha_{23} = 0$

we obtain

$$\overline{F_2}(y) = \begin{pmatrix} 0 \\ \alpha_{21}y_1^2 + (\alpha_{22} - 2d)y_1y_2 \end{pmatrix} \tag{41}$$

And thus (35) takes the form

$$y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_{21}y_1^2 + (\alpha_{22} - 2d)y_1y_2 \end{pmatrix} + O(|y|)^3 \tag{42}$$

This is the required normal form of Bogdanov-Takens (codimension 2) bifurcation if the coefficients satisfy the following conditions:

$$\alpha_{21} \neq 0 \tag{43}$$

$$\alpha_{22} - 2d \neq 0 \tag{44}$$

Where

$$d = -\alpha_{12} = -\eta(1 - \theta - 3x_1) + \frac{\alpha^2\theta}{(\theta + x_1)^2}$$

$$\alpha_{21} = \frac{\alpha^2\theta\eta x_1}{(\theta + x_1)^2}(1 - \theta - 3x_1) - \frac{\alpha^4\theta^2 x_1}{(\theta + x_1)^4} - \left(\frac{\alpha(\theta - x_1)}{\eta(\theta + x_1)} - 1\right)\frac{\alpha^2\theta x_1}{(\theta + x_1)^2} + \frac{\mu\alpha^3\theta^2 x_1}{(\theta + x_1)^3}$$

$$\alpha_{22}-2d=\frac{\alpha(\theta-x_1)}{\eta(\theta+x_1)}-\frac{2\mu\alpha\theta}{\theta+x_1}-1+\ 2\eta(1-\theta-3x_1)-\frac{\alpha^2\theta}{(\theta+x_1)^2}$$

Thus we have the following result:

Theorem1:

The system (3, 4) experiences Bogdanov-Takens (codimension 2) bifurcation at an equilibrium point (x_1, y_1) *if the involved parameters satisfy the conditions (13), (14), (43) and (44).*

Now we shall use Theorem 1 to establish the occurrence of Bogdanov-Takens bifurcation in the system (3, 4) at the points $e=\left(\frac{1-\sqrt{\eta}}{2},\frac{1+\sqrt{\eta}}{2\sqrt{\eta}}\right)$and $f=\left(\frac{1-2\theta}{3},\frac{\alpha(1-2\theta)}{\eta(\theta+1)}\right)$.

Theorem 2(Bogdanov-Takens bifurcation point):

If$\boldsymbol{\alpha=\theta=\frac{\sqrt{\eta}}{2}, 0<\eta}<1$*and*$\boldsymbol{\mu=\frac{\eta(1-\sqrt{\eta})}{1+\sqrt{\eta}}}$*then the interior equilibrium point*$(\boldsymbol{x_1},\boldsymbol{y_1})=\left(\boldsymbol{\frac{1-\sqrt{\eta}}{2},\frac{1+\sqrt{\eta}}{2\sqrt{\eta}}}\right)$**.**

Proof: to establish the theorem it is sufficient to verify the conditions (43) and (44). In view of $\mathbf{0}<\eta<1$, it can be easily verified that

$$\boldsymbol{\alpha_{21}=\frac{\sqrt{\eta}}{8}(1-\sqrt{\eta})\left[\sqrt{\eta}(\eta^2-1)+\eta^2(\mu-1)\right]}<0 \tag{45}$$

$$\boldsymbol{\alpha_{22}-2d=\frac{1}{\eta}\left(\frac{\sqrt{\eta}}{2}(3\eta^2-1)-\eta^2(\mu+1)\right)}<0$$

(46)

Which shows $\boldsymbol{\alpha_{21}\neq 0}$ and $\boldsymbol{\alpha_{22}-2d}$ and hence the proof.

Theorem 3 (cusp point):

If $\boldsymbol{0<\alpha<\frac{1}{2}, 0<\theta<\frac{1}{2}, 0<\frac{\alpha\theta}{\eta}<1/12}$ *and*$\boldsymbol{\mu=\frac{9\alpha^2\theta(1-2\theta)}{(\theta+1)^2(\eta(\theta+1)+\alpha(1-2\theta))}}$ *then the interior equilibrium point* $(\boldsymbol{x},\boldsymbol{y})=\left(\boldsymbol{\frac{1-2\theta}{3},\frac{\alpha(1-2\theta)}{\eta(\theta+1)}}\right)$*is a cusp point of Codimension 2.*

Proof: To prove the theorem it is sufficient to verify (60) and (61) using the relation given (13) and (14). Now let us take

$$\alpha_{21}=\frac{\alpha^2\theta\eta x_1}{(\theta+x_1)^2}(1-\theta-3x_1)-\frac{\alpha^4\theta^2x_1}{(\theta+x_1)^4}-\left(\frac{\alpha(\theta-x_1)}{\eta(\theta+x_1)}-1\right)\frac{\alpha^2\theta x_1}{(\theta+x_1)^2}+\frac{\mu\alpha^3\theta^2x_1}{(\theta+x_1)^3}.$$

Upon some long but simple simplifications we obtain the following

$$\alpha_{21}=\frac{\alpha^2\theta\eta x_1}{(\theta+x_1)^2}(1-\theta-3x_1)-\frac{\alpha^4\theta^2x_1}{(\theta+x_1)^4}-\left(\frac{\alpha(\theta-x_1)}{\eta(\theta+x_1)}-1\right)\frac{\alpha^2\theta x_1}{(\theta+x_1)^2}+\frac{\mu\alpha^3\theta^2x_1}{(\theta+x_1)^3}.$$

$$>\frac{3\alpha^2\theta(1-2\theta)}{(\theta+1)^2}\left[1-\frac{12\alpha\theta}{\eta}+\eta\theta+\frac{\alpha(1-2\theta)}{\eta(\theta+1)}+\frac{27\alpha^2\theta(1-2\theta)}{(\theta+1)^3(\eta(\theta+1)+\alpha(1-2\theta))}\right]$$

$$=\frac{3\alpha^2\theta(1-2\theta)}{(\theta+1)^2}\left[\left(\frac{1}{12}-\frac{\alpha\theta}{\eta}\right)+\eta\theta+\frac{\alpha(1-2\theta)}{\eta(\theta+1)}+\frac{27\alpha^2\theta(1-2\theta)}{(\theta+1)^3(\eta(\theta+1)+\alpha(1-2\theta))}\right]$$

Observe that $\alpha_{21}>0$ under the assumptions given in the theorem. And now let us simplify (44)

$$\alpha_{22}-2d=\frac{\alpha(\theta-x_1)}{\eta(\theta+x_1)}-\frac{2\mu\alpha\theta}{\theta+x_1}-1+\ 2\eta(1-\theta-3x_1)-\frac{\alpha^2\theta}{(\theta+x_1)^2}$$

Again upon some long but simple simplifications we obtain the following

$$\alpha_{22} - 2d = \frac{\alpha(\theta - x_1)}{\eta(\theta + x_1)} - \frac{2\mu\alpha\theta}{\theta + x_1} - 1 + 2\eta(1 - \theta - 3x_1) - \frac{\alpha^2\theta}{(\theta + x_1)^2}$$

$$< \frac{12\alpha\theta}{\eta} - 1 - \frac{\alpha(1-2\theta)}{\eta(\theta+1)} - \frac{2}{3}\eta(1-2\theta) - \frac{54\alpha^3\theta^2(1-2\theta)}{(\theta+1)^3(\eta(\theta+1)+\alpha(1-2\theta))}$$

$$= 12\left(\frac{\alpha\theta}{\eta} - \frac{1}{12}\right) - \frac{\alpha(1-2\theta)}{\eta(\theta+1)} - \frac{2}{3}\eta(1-2\theta) - \frac{54\alpha^3\theta^2(1-2\theta)}{(\theta+1)^3(\eta(\theta+1)+\alpha(1-2\theta))}$$

Here also observe that $\alpha_{22} - 2d < 0$ under the assumption given in the theorem. That is both $\alpha_{21} \neq 0$ and $\alpha_{22} - 2d \neq 0$ and hence the proof.

Funding: This study received no specific financial support.

Competing Interests: The authors declare that they have no conflict of interests.

Contributors/Acknowledgement: All authors contributed equally to the conception and design of the study. I would like to express my sincere gratitude to Dr. P.D.N. Srinivasu, Professor of Mathematics for his willingness to see and comment all my works, his many valuable suggestions, encouragement and constant support in doing this work. I also express my greatest appreciation to the reviewers for their constructive comments.

REFERENCES

[1] N. Ali, *Nonlinear dynamical systems and chaos*. Englewood Cliffs: Prentice Hall, 1983.

[2] Temesgen Tibebu Mekonnen, "Bifurcation analysis on the dynamics of a generalist predator pre system," *International Journal of Ecosystem, USA*, vol. 2, pp. 38-43, 2012.

[3] F. George Simmons, *Differential equations*. New Delhi: McGraw-HILL Publishing Company Ltd, 1972.

[4] H. W. Turnbull, *Theory of equations*. New York: Intercience Publishers Inc, 1952.

[5] P. Lawrence, *Differential equations and dynamical systems*, 3rd ed. New York: Springer. Available: http://journal.sapub.org/ajb, 2001.

[6] A. Yuri Kuzentsov, *Elements of applied bifurcation theory*, 2nd ed. New York: Springer, 1998.