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ON THE COMPONENT ANALYSIS AND TRANSFORMATION OF AN EXPLICI FIFTH – STAGE FOURTH – ORDER RUNGE – KUTTA METHODS

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ABSTRACT

This work is designed to transform the fifth stage – fourth order explicit Runge-Kutta method with the aim of projecting a new method of implementing it through tree diagram analysis. Efforts will be made to represent the equations derived from the y derivatives and x,y derivatives separately on Butcher's rooted trees. This is because the rooted trees and derived equations for the y derivatives and x,y derivatives are the same for the explicit fourth-stage fourth-order methods, hence, we are motivated to analyze the fifth-stage fourth-order method. This idea is also derivable from general graphs and combinatorics.

Keywords: Rooted tree diagram, Transformation, Vertex, Explicit, Y partial derivatives, X,Y partial derivatives, Linear and non- linear equations, Taylor series, Graphs.

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1. INTRODUCTION

Explicit Runge– kutta (ERK) formulas are among the oldest and best – understood schemes in the numerical analysis tool kit. However, according to Byrne and Hindmarsh [1] "despite the evolutions of a vast and comprehensive body of knowledge, ERK algorithms continue to be sources of active research". The history of ERK methods began almost a century ago. Classic references are Heun [2]; Kutta [3] and Runge [4]. According to Lambert [5] the Runge–Kutta methods represent an important family of implicit and explicit iterative methods for approximation of ordinary differential equations in numerical analysis.

Because of their elegance and simplicity, ERK methods are usually among the first to be taught in the ODE section of a numerical methods course. Thankfully, good quality introductory texts no longer dismiss "the Runge – kutta method" as a fixed step size implementation of the classic 4^{th} order ERK formula. However, significant advancements in the state – of – the – art which post – date the work of Fehlberg [6] even in the fundamental area of deriving ERK

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formulas, tend to be ignored. According to Kahaner, et al. [7] another side effect of the simple nature of the ERK formula is that a generation of non – experts have been tempted to write their own "quick and dirty" codes. It is widely acknowledged that "squeaky – clean" codes require a great deal of expertise and programming effort. A high – level discussion of some of the issues involved in ERK is found in the work of Shampine and Gladwell [8]. Recent work on Runge-Kutta analysis include Agbeboh [9]; Agbeboh, et al. [10]; Esekhaigbe [11] and Butcher [12]. More recent works are that of Van Der Houwen and Sommeijer [13]; Van Der Houwen and Sommeijer [14]; Van Der Houwen and Sommeijer [15].

An extensive variation of a fourth order Runge-Kutta Method can be seen in Agbeboh, et al. [10]; Agbeboh and Ehiemua [16]; Agbeboh [17].

The work of Butcher [18]; Butcher [19]; Butcher [12]; Butcher [20] revealed much success in the analysis of explicit Runge-Kutta methods and their transformation to rooted tree diagrams. This was because the continuation of the process of Taylor Series gives rise to very complicated formulas. It was therefore, advantages to use a graphical representation for a convenient analysis of the order of a Runge – kutta method; hence, the basic tree theory was introduced. A tree is a rooted graph which contains no circuits. The symbol T is used to represent the tree with only one vertex. All rooted trees can be represented using T and the operation $[t_1, t_2, \ldots, t_m]$.Hence, it is the differentials and equations derived that are represented on trees so as to enable us compare the order condition with their differentials for varying parameters. Connes and Kreimer [21] pointed out that the Butcher group is the group of characters that had risen independently in their own work on rooted trees analysis in Runge – kutta methods for solving initial-value problems in ordinary differential equations. Recent works on rooted tree analysis include Butcher [22]; Butcher [23]; Butcher [24]; Brouder [25]; Brouder [26] e.t.c. Our work on the component analysis and transformation of an explicit fourth-stage fourth-order Runge-Kutta method revealed the fact that the fifth-stage fourth-order method can still be researched. See Agbeboh and Esekhaigbe [27].

Conclusively, despite the fact that good, reliable explicit Runge-Kutta formulas exist, there is still need for their transformation to rooted tree diagrams. Traditionally, Runge – kutta methods are all explicit, although recently, implicit Runge – kutta methods, which have improved weak stability characteristics have been considered. However, the transformation of implicit Runge-Kutta methods to rooted tree diagrams can also be explored.

1.1. Definition of Terms

1. A Runge – Kutta method is said to be A- stable if its stability region contains ¢, the non – positive half – plane i.e.$|r(z)| \leq 1$, for all z ϵ¢.

2. The condition $\sum_{j=i}^{s} b_j = 1$, is known as the consistency and convergence condition which is also necessary and sufficient for the local truncation error of the method to have asymptotic behavior. Hence, $\emptyset(x, y, 0) = f(x, y)$
3. A Runge – Kutta method has stability region when the region $|r(z)| \leq 1$ and a strict stability regions when the region $|r(z)| < 1$
4. A Runge – Kutta method is said to possess stability properties if properties possessed by the exact solution are as boundedness and convergence to zero of its numerical solutions.
5. A Runge – Kutta method is L – Stable if the region $|r(z)| \leq 1$ for all z $\epsilon ¢$ and in addition that $\lim_{|z|\to\infty} |r(z)| = 0$. This property was first studied by Ehle [28]
6. A Runge – Kutta method is said to be AN – Stable if for $z_1, z_2 \ldots z_s \epsilon ¢$ such that $z_i = z_j$ if $c_i = c_j$, $\det(1 - Az)$, For $|r(z)| \leq 1$, R (z) – 1 + $b^T z(1 - Az)^{-1}$ e.
   The name AN – Stability in this definition comes from the observation that it is A – stability modified for non – autonomous problems.
7. If a non – confluent Runge – Kutta method is AN – Stable, then $M = BA + A^T B - bb^T$ and $B = \text{ding}(b_1, b_2 \ldots b_s)$ are each positive semi definite matrices.
8. A Runge–Kutta method is said to be B – Stable if for any problem $y^1(x) = f(y(x))$ where f is such that $(f(u) - f(v))T(u - v) \leq 0$ for all u and v, then $||y_n - z_n|| \leq ||y_{n-1} - y_{n-1}||$ for the solution sequences . . ., y_{n-1}, y_n, . . . and . . .z_{n-1}, z_n, ...
9. According to Lambert [5] the traditional criterion for ensuring that a numerical method is stable or called absolutely stable" is subject to the linear test equation.
   Y' λy; λ ϵC; Re (λ) < 0, where λ is complex according to Butcher [19] the test equation can further be reduced to $y_{n-1} = R(\lambda h)$,
   $R(\lambda h)$ is called the stability polynomial

2. METHODS OF DERIVATION

i. From the general Runge-Kutta method, get a Fifth Stage-Fourth order method
ii. Obtain the Taylor series expansion of $k_{i's}^{;}$ about the point (x_n, y_n), i=2,3,4,5
iii. Carry out substitution to ensure that all the $k_{i's}^{;}$ are in terms of k_1 only.
iv. Insert the $k_{i's}^{;}$ in terms of k_1 only into $b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5$

Separate all partial derivatives involving only y with their coefficient from all partial derivatives involving x, y and their coefficients.

v. Compare the coefficients of all partial derivatives involving only y with Taylor series expansion involving only partial derivatives with respect to y of the form:

$$\emptyset_T(x, y, h) = f + \frac{h}{2!} f f_y + \frac{h^2}{3!}\left(f f_y^2 + f^2 f_{yy}\right) + \frac{h^3}{4!}\left(4f^2 f_y f_{yy} + f f_y^3 + f^3 f_{yyy}\right) + \frac{h^4}{5!}\left(7f^3 f_y f_{yyy} + 4f^3 f_{yy}^2 + 11 f^2 f_y^2 f_{yy} + f f_y^4 + f^4 f_{yyyy}\right)$$

vi. As a result, a set of linear/non linear equations will be generated. Represent these equations and their partial derivatives on Butcher's rooted tree diagram.

vii. Compare the coefficient of all partial derivatives involving x, y only with Taylor series expansion involving partial derivative of x, y only of the form:

$$\phi(x,y,h) = f + \frac{h}{2!}f_x + \frac{h^2}{3!}\left(f_{xx} + 2ff_{xy} + f_xf_y\right) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + f_xf_y^2)$$

viii. As a result also, a set of linear/non-linear equations will be generated. Represent those equations and their x, y partial derivatives on Butcher's rooted tree diagram.

ix. Vary the parameters from the set of equations generated above. A new fifth-stage fourth-order explicit Runge-Kutta formula will be birthed.

2.1 Derivation of the Fifth – Stage Fourth-Order ERK Method

According to Lambert [5] the general R – Stage Runge – Kutta method is:

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$
$$\phi(x_n, y_n, h) = \textstyle\sum_{r=1}^{R} b_r k_r$$
$$k_1 = f(x,y)$$
$$k_r = f\left(x + hc_r, y + h\sum_{s=1}^{r-1} a_{rs}k_s\right), r = 2, 3, \ldots R$$

The formula is defined by the number of stages s, the nodes $[c_r]^{s}{}_{r=1}$, the internal weights $[a_{rs}]^{r-1,s}{}_{s=1,r=2}$ and the external weights $[b_r]^{s}{}_{r=1}$.

From the above scheme, the fifth stage fourth – order method is:

$$y_{n+1} = y_n + h\,(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5)$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + c_2, y_n + ha_{21}k_1)$$
$$k_3 = f(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2))$$
$$k_4 = f(x_n + c_4h, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3))$$
$$k_5 = f(x_n + c_5h, y_n + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4))$$

Using Taylor's series expansion for $k_i's$, we have:

$$k_1 = f(x_n, y_n)$$
$$k_2 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_2h\frac{d}{dx} + ha_{21}k_1\frac{d}{dy})^r f(x_n, y_n)$$
$$k_3 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_3h\frac{d}{dx} + h(a_{31}k_1 + a_{32}k_2)\frac{d}{dy})^r f(x_n, y_n)$$
$$k_4 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_4h\frac{d}{dx} + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\frac{d}{dy})^r f(x_n, y_n)$$

$$k_5 = \sum_{r=o}^{\infty} \frac{1}{r!}(c_5 h \frac{d}{dx} + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)\frac{d}{dy})^r f(x_n, y_n)$$

Hence, we have:

$$k_1 = f$$

$$k_2 = f + \left(c_2hf_x + ha_{21}k_1f_y\right) + \frac{1}{2!}\left(c_2hf_x + ha_{21}k_1f_y\right)^2 + \frac{1}{3!}\left(c_2hf_x + ha_{21}k_1f_y\right)^3 + \frac{1}{4!}\left(c_2hf_x + ha_{21}k_1f_y\right)^4 + 0(h^5)$$

$$k_3 = f + \left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right) + \frac{1}{2!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^2 + \frac{1}{3!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^3 + \frac{1}{4!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^4 + 0(h^5)$$

$$k_4 = f + \left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right) + \frac{1}{2!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^2 + \frac{1}{3!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^3 + \frac{1}{4!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^4 + 0(h^5)$$

$$k_5 = f + \left(c_5hf_x + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)\,f_y\right) + \frac{1}{2!}\left(c_5hf_x + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)\,f_y\right)^2 + \frac{1}{3!}\left(c_5hf_x + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)\,f_y\right)^3 + \frac{1}{4!}\left(c_5hf_x + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)\,f_y\right)^4 + 0(h^5)$$

Expanding fully and substituting the various k_i' s, i = 2, 3, 4, 5 into their various positions in terms of k_1 only and collecting like terms, in terms of y derivatives and (x, y) derivatives separately, we have:

$$k_1 = f$$

$$k_2 = f + ha_{21}ff_y + \frac{h^2}{2!}a_{21}^2f^2f_{yy} + \frac{h^3}{3!}a_{21}^3f^3f_{yyy} + \frac{h^4}{4!}a_{21}^4f^4f_{yyyy} + hc_2f_x + \frac{h^2}{2!}c_2^2f_{xx} + h^2c_2a_{21}ff_{xy}\frac{h^3}{3!}c_2^3f_{xxx} + \frac{h^3}{2!}c_2^2a_{21}ff_{xxy} + \frac{h^3}{2!}c_2a_{21}^2f^2f_{xyy} + \frac{h^4}{4!}c_2^4f_{xxxx} + \frac{h^4}{3!}c_2^3a_{21}ff_{xxxy}\frac{h^4}{2!\,2!}c_2^2a_{21}^2f^2f_{xxyy} + \frac{h^4}{3!}c_2a_{21}^3f^3f_{xyyy} + 0\,(h^5)$$

$$k_3 = f + h(a_{31} + a_{32})ffy + h^2 a_{21}a_{32}ff_y^2 + \frac{h^2}{2!}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2 f_{yy}$$
$$+ \frac{h^3}{3!}a_{21}a_{32}\big(a_{21} + 2(a_{31} + a_{32})\big)f^2 f_y f_{yy}$$
$$+ \frac{h^3}{3!}(a_{31}^3 + 3a_{31}^2 a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3 f_{yyy}$$
$$+ \frac{h^4}{3!}(a_{32}a_{21}^3 + 3a_{31}^2 a_{32}a_{21} + 3a_{32}^3 a_{21} + 6a_{31}a_{32}^2 a_{21})f^3 f_y f_{yyy}$$
$$+ \frac{h^4}{2!}a_{21}^2 a_{32}(a_{31} + a_{32})f^3 f_{yy}^2 + \frac{h^4}{2!}a_{32}^2 a_{21}^2 f^2 f_y^2 f_{yy}$$
$$+ \frac{h^4}{4!}(a_{31}^4 + 4a_{31}^3 a_{32} + 6a_{31}^2 a_{32}^2 + 4a_{31}a_{32}^3 + a_{32}^4)f^4 f_{yyyy} + hc_3 f_x + \frac{h^2}{2!}c_3^2 f_{xx}$$
$$+ h^2 c_3(a_{31} + a_{32})ff_{xy} + h^2 c_2 a_{32}f_x f_y + \frac{h^3}{3!}c_3^3 f_{xxx} + \frac{h^3}{2!}c_3^2(a_{31} + a_{32})ff_{xxy}$$
$$+ \frac{h^3}{2!}c_3(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2 f_{xyy} + h^3 c_2 a_{32}(a_{31} + a_{32})ff_x f_{yy}$$
$$+ h^3 a_{21}a_{32}(c_2 + c_3)ff_y f_{xy} + \frac{h^3}{2!}c_2^2 a_{32}f_y f_{xx} + h^3 c_2 c_3 a_{32}f_x f_{xy} + \frac{h^4}{4!}c_3^4 f_{xxxx}$$
$$+ \frac{h^4}{3!}c_2^3 a_{32}f_{xxx}f_y + \frac{h^4}{2!}c_3^2 c_2 a_{32}f_x f_{xxy} + \frac{h^4}{2!}a_{21}a_{32}(c_2^2 + c_3^2)ff_y f_{xxy}$$
$$+ \frac{h^4}{3!}a_{21}a_{32}(2c_2 a_{31} + 3c_2 a_{21} + 6c_3 a_{31})f^2 f_y f_{xyy} + \frac{h^4}{2!}c_3 a_{32}c_2^2 f_{xx}f_{xy}$$
$$+ \frac{h^4}{2!}c_2^2 a_{32}(a_{31} + a_{32})ff_{xx}f_{yy}$$
$$+ \frac{h^4}{2!}a_{21}a_{32}(2c_2 a_{31} + 2c_2 a_{32} + c_3 a_{21})f^2 f_{xy}f_{yy} + h^4 c_3 a_{32}c_2 a_{21}ff_{xy}^2$$
$$+ \frac{h^4}{2!}a_{32}^2 c_2^2 f_x^2 f_{yy} + h^4 a_{32}^2 a_{21}c_2 ff_x f_y f_{yy} + \frac{h^4}{2!}c_3 c_2 a_{32}(6a_{31} + 2a_{32})ff_x f_{xyy}$$
$$+ \frac{h^4}{2!}c_2 a_{32}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2 f_x f_{yyy} + \frac{h^4}{3!}c_3^3(a_{31} + a_{32})ff_{xxxy}$$
$$+ \frac{h^4}{2!\,2!}c_3^2(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2 f_{xxyy}$$
$$+ \frac{h^4}{3!}c_3(a_{31}^3 + 3a_{31}^2 a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3 f_{xyyy} + 0\,(h^5)$$
$$+12a_{41}a_{42}^2 a_{43} + 2a_{41}a_{42}a_{43}^2$$

$+ 6a_{41}^2a_{43}^2+ 4a_{42}^3a_{43}+ 4a_{41}a_{42}^3+ a_{42}^4+ a_{43}^4) f^4f_{yyyy} + \mathrm{h}c_4f_x + h^2(c_4a_{42} + c_3a_{43})f_xf_y + \frac{h^2}{2!}c_4^2f_{xx}$

$+ h^2c_4(a_{41} + a_{42} + a_{43})ff_{xy} + \frac{h^3}{2!}(c_2^2a_{42} + c_3^2a_{43})f_{xx}f_y + h^2(c_4a_{21}a_{42}+ c_3a_{31}a_{43}+ c_3a_{32}a_{43})ff_{xy}f_y + h^3c_2a_{32}a_{43}f_xf_y^2+ h^3(c_2c_4a_{42}+ c_3c_4a_{43}) f_xf_{xy}$
$+h^3c_2a_{32}a_{43}f_xf_y^2+ h^3(c_2c_4a_{42}+ c_3c_4a_{43}) f_xf_{xy} + h^3(c_2a_{21}a_{42} + c_4a_{31}a_{43}+ c_4a_{32}a_{43})ff_yf_{xy}+ h^3(c_2a_{41}a_{42}+ c_3a_{41}a_{43}+ c_3a_{42}a_{43}+ c_2a_{42}a_{43}+ c_2a_{42}^2 + c_3a_{43}^2)ff_xf_{yy} +$
$\frac{h^3}{3!}c_4^3f_{xxx} + \frac{h^3}{2!}(c_4^2a_{41} + c_4^2a_{42}+ c_4^2a_{43})ff_{xxy}+ \frac{h^3}{2!}c_4(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + a_{42}^2 +$

$2a_{42}a_{43} + a_{43}^2)f^2f_{xyy} + \frac{h^4}{3!}(c_2^3a_{42} + c_3^3a_{43})f_{xxx}f_y + \frac{4}{3!}(3c_2^2a_{21}a_{42}+c_3^2a_{31}a_{43}+3c_3^2a_{32}a_{43} +$

$3c_4^2a_{21}a_{42} + 3c_4^2a_{31}a_{43}+ 3c_4^2a_{32}a_{43})ff_{xxy}f_y + \frac{h^4}{2!}(c_2a_{21}^2a_{42}+$
$2c_3a_{31}a_{32}a_{43} + c_3a_{31}^2a_{43}+c_3a_{32}^2a_{43}+ 2c_4a_{21}a_{41}a_{42}+ 2c_4a_{31}a_{41}a_{43}+ 2c_4a_{32}a_{41}a_{43}+$
$2c_4a_{21}a_{42}^2+ 2c_4a_{31}a_{42}a_{43} + 2c_4a_{32}a_{42}a_{43} + 2c_4a_{21}a_{42}a_{43}+ 2c_4a_{31}a_{43}^2+ 2c_4a_{32}a_{43}^2)f^2f_yf_{xyy}+$
$\frac{h^4}{3!}(c_2^2a_{32}a_{43})f_{xx}f_y^2 + h^4(c_2a_{21}a_{32}a_{43}+ c_3a_{21}a_{32}a_{43}+ c_4a_{21}a_{32}a_{43})ff_{xy}f_y^2 +$

$\frac{h^4}{2!}(2c_2a_{31}a_{32}a_{43} + 2c_2a_{32}^2a_{43} + 2c_2a_{32}a_{41}a_{43}+ 2c_2a_{32}a_{42}a_{43} + 2c_2a_{31}a_{42}a_{43} +$
$2c_2a_{32}a_{42}a_{43} + 2c_3a_{21}a_{42}a_{43}+ 2c_2a_{21}a_{42}^2 + c_2a_{32}a_{43}^2 + c_3a_{31}a_{43}^2+ c_3a_{32}a_{43}^2+ c_3a_{31}a_{43}^2+$
$c_3a_{32}a_{43}^2)ff_xf_yf_{yy} + h^4(c_2c_3a_{32}a_{43} + c_2c_4a_{32}a_{43})f_xf_yf_{xy} + \frac{h^4}{2!}(c_2^2c_4a_{42} + c_3^2c_4a_{43})f_{xx}f_{xy}+$
$h^4(c_2c_4a_{21}a_{42} + c_3c_4a_{31}a_{43} + c_3c_4a_{32}a_{43})$
$ff_{xy}^2 + \frac{h^4}{2!}(c_4a_{21}^2a_{42}+ c_4a_{31}^2a_{43}+ 2c_4a_{31}a_{32}a_{43}+ c_4a_{32}^2a_{43} + 2c_2a_{21}a_{41}a_{42}+ 2c_3a_{31}a_{41}a_{43}+$
$2c_3a_{32}a_{41}a_{43}+ 2c_3a_{31}a_{42}a_{43}+ 2c_3a_{32}a_{42}a_{43}+ c_2a_{21}a_{42}^2+ c_3a_{31}a_{43}^2 + c_3a_{32}a_{43}^2) f^2f_{xy}f_{yy} +$
$\frac{h^4}{2!}(2c_2c_3a_{42}a_{43}+ c_2^2a_{42}^2+ c_3^2a_{43}^2)f_x^2f_{yy} + \frac{h^4}{2!}(c_2c_4^2a_{42} + c_3c_4^2a_{43})f_xf_{xxy} + h^4(c_2c_4a_{41}a_{42} +$

$c_3c_4a_{41}a_{43} + c_2c_4a_{42}^2 + c_3c_4a_{42}a_{43}) + c_2c_4a_{42}a_{43}+ c_3c_4a_{43}^2)ff_xf_{xyy}+ \frac{h^4}{2!}(c_2c_{41}^2a_{42} +$
$c_3c_{41}^2a_{43}+ 2c_2a_{41}a_{42}^2 + 2c_3a_{41}a_{42}a_{43} + 2c_2a_{41}a_{42}a_{43}+ c_3c_{42}^2a_{43} + 2c_2c_{42}^2a_{43} + 2c_3a_{41}a_{43}^2 +$
$2c_3a_{42}a_{43}^2 + c_2a_{42}a_{43}^2+ c_2c_{42}^3+ c_3a_{43}^3)f^2f_xf_{yyy} + \frac{h^4}{4!}c_4^4f_{xxxx}+ \frac{h^4}{3!}(c_4^3a_{41} + c_4^3a_{42} +$

$c_4^3a_{43})ff_{xxxy} + \frac{h^4}{2!2!}c_4^2(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43}+ a_{42}^2+ 2a_{42}a_{43}+ a_{43}^2)f^2f_{xxyy}+ \frac{h^4}{3!}c_4(a_{41}^3 +$
$3a_{41}^2a_{42} + 3a_{41}^2a_{43} + 3a_{41}a_{42}^2+ 6a_{41}a_{42}a_{43}+ 3a_{42}^2a_{43}+ 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3) f^3f_{xyyy}+$
$\frac{h^4}{2!2!}(2c_2^2a_{41}a_{42} + 2c_3^2a_{41}a_{43} + 2c_3^2a_{42}a_{43}+ c_2^2a_{42}^2 + c_3^2a_{43}^2)ff_{xx}f_{yy.} + 0(h^5).$

$$k_5 = f + h(a_{51} + a_{51} + a_{53} + a_{54})ffy$$
$$+\ h^2(a_{21}a_{51} + a_{31}a_{53} + a_{32}a_{53} + a_{41}a_{54} + a_{42}a_{54} + a_{43}a_{54})ff_y^2$$
$$+\frac{h^2}{2!}(a_{51}^2 + 2a_{51}a_{52} + 2a_{51}a_{53} + 2a_{51}a_{54} + a_{52}^2 + 2a_{52}a_{53} + 2a_{52}a_{54} + a_{53}^2$$
$$+ 2a_{53}a_{54} + a_{54}^2)f^2f_{yy} + \frac{h^3}{2!}(a_{21}^2a_{52} + a_{31}^2a_{53} + 2a_{31}a_{32}a_{53} + a_{32}^2a_{53}$$
$$+ a_{41}^2a_{54} + +2a_{41}a_{42}a_{54} + 2a_{41}a_{42}a_{54} + 2a_{41}a_{43}a_{54} + 2a_{42}a_{43}a_{54} + a_{42}^2a_{54}$$
$$+ a_{43}^2a_{54} + 2a_{21}a_{51}a_{52} + 2a_{31}a_{51}a_{53} + 2a_{32}a_{51}a_{53} + 2a_{41}a_{51}a_{54}$$
$$+ 2a_{42}a_{51}a_{54} + 2a_{43}a_{51}a_{54} + 2a_{21}a_{52}^2 + 2a_{21}a_{52}a_{53} + 2a_{31}a_{52}a_{53}$$
$$+ 2a_{32}a_{52}a_{53} + 2a_{41}a_{52}a_{54} + 2a_{42}a_{52}a_{54} + 2a_{43}a_{52}a_{54} + 2a_{21}a_{52}a_{54}$$
$$+ 2a_{31}a_{53}^2 + 2a_{32}a_{53}^2 + 2a_{41}a_{53}a_{54} + 2a_{42}a_{53}a_{54} + 2a_{43}a_{53}a_{54}$$
$$+ 2a_{31}a_{53}a_{54} + 2a_{32}a_{53}a_{54} + 2a_{41}a_{54}^2 + 2a_{42}a_{54}^2 + 2a_{43}a_{54}^2)\ f^2f_yf_{yy}$$
$$+ h^3(a_{32}a_{21}a_{53} + a_{21}a_{42}a_{54} + a_{31}a_{43}a_{54} + a_{32}a_{43}a_{54})ff_y^2$$
$$+\frac{h^3}{3!}(a_{51}^3 + 3a_{51}^2a_{52} + 3a_{51}^2a_{53} + 3a_{51}^2a_{54} + 3a_{51}a_{52}^2 + 6a_{51}a_{52}a_{53}$$
$$+ 6a_{51}a_{52}a_{54} + 3a_{51}a_{53}^2 + 6a_{51}a_{53}a_{54} + 3a_{51}a_{52}^2 + a_{52}^3 + 3a_{52}^2a_{53}$$
$$+ 3a_{52}^2a_{54}3a_{52}a_{53}^2 + 6a_{52}a_{53}a_{54} + 3a_{52}a_{54}^2 + a_{53}^3 + 3a_{53}^2a_{54} + 3a_{53}a_{54}^2$$
$$+ a_{54}^3)f^3f_{yyy} + hc_5f_x + h^2(c_2a_{52} + c_3a_{53} + c_4a_{54})f_xf_y + \frac{h^2}{2!}c_5^2f_{xx}$$
$$+ h^2(c_5a_{51} + c_5a_{52} + c_5a_{53} + c_5a_{54})ff_{xy} + \frac{\mathrm{h}^3}{2!}(\mathrm{c}_5^2\mathrm{a}_{52} + \mathrm{c}_3^2\mathrm{a}_{53} + \mathrm{c}_4^2\mathrm{a}_{54})\mathrm{f}_{\mathrm{xx}}\mathrm{f}_{\mathrm{y}}$$
$$+ \mathrm{h}^3(c_2a_{21}a_{52} + c_3a_{31}a_{53} + c_3a_{32}a_{53} + a_4a_{41}a_{54} + a_4a_{42}a_{54}$$
$$+ a_4a_{43}a_{54})ff_{xy}f_y + h^3(a_{32}c_2a_{53} + c_2a_{42}a_{54} + c_3a_{43}a_{54})f_xf_y^2$$
$$+ h^3(c_2c_5a_{52} + c_3c_5a_{53} + c_4c_5a_{54})f_xf_xy$$
$$+ h^3(a_{21}c_5a_{52} + a_{31}c_5a_{53} + a_{32}c_{53} + a_{41}c_5a_{54} + a_{42}c_5a_{54} + a_{43}c_5a_{54})ff_yf_{xy}$$
$$+ h^3(c_2a_{51}a_{52} + c_3a_{51}a_{53} + c_4a_{51}a_{54} + c_2a_{52}^2 + c_2a_{52}a_{53} + c_3a_{52}a_{53}$$
$$+ c_4a_{52}a_{54} + c_2a_{52}a_{54} + c_3a_{53}^2 + c_4a_{53}a_{54} + c_3a_{54}a_{53} + c_4a_{54}^2)ff_xff_y$$
$$+\frac{h^3}{3!}c_5^3f_{xxx} + \frac{h^3}{2!}(c_5^2a_{51} + c_5^2a_{52} + c_5^2a_{53} + c_5^2a_{54})ff_{xxy} + \frac{h^3}{2!}(c_5a_{51}^2$$
$$+ 2c_5a_{51}a_{52} + 2c_5a_{51}a_{53} + 2c_5a_{51}a_{54} + c_5a_{52}^2 + 2c_5a_{52}a_{53} + 2c_5a_{52}a_{54}$$
$$+ c_5a_{53}^2 + 2c_5a_{53}a_{54} + c_5a_{54}^2)f^2f_{xyy}$$

Putting the $k_{i's}^{i}$ *(y derivatives only) into*$y_{n+1} = y_n + h(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5$ *where*$\phi(x,y,h) = b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5$ *and equating coefficients with the Taylor series expansion:*

$\emptyset_T(x,y,h) = f + \frac{h}{2!} f f_y + \frac{h^2}{3!}(f f_y^2 + f^2 f_{yy}) + \frac{h^3}{4!}(4 f_y^2 f_y f_{yy} + f f_y^3 + f^3 f_{yyy}) + \frac{h^4}{5!}(4 f^3 f_y f_{yyy} + 4 f^3 f_{yy}^2 + 11 f^2 f^2 f_{yy} + f f_y^4 + f^4 f_{yyyy})$, we have the following equations:

$$b_1 + b_2 + b_3 + b_4 + b_5 = 1 \quad (1)$$

$$b_2 c_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 = 1/2 \quad (2)$$

$$b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 + b_5 c_5^2 = 1/3 \quad (3)$$

$$b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 + b_5 c_5^3 = 1/4 \quad (4)$$

$$b_3 c_2 a_{32} + b_{43} c_2 a_{42} + b_4 c_3 a_{43} + b_5 c_2 a_{52} + b_5 c_3 a_{53} + b_5 c_4 a_{54} = 1/6 \quad (5)$$

$$b_3 a_{32} c_2 c_3 + b_4 a_{42} c_2 c_4 + b_4 a_{43} c_3 c_4 + b_5 c_2 c_5 a_{52} + b_5 a_{53} c_3 c_5 + b_5 c_4 c_5 a_{54} = 1/8 \quad (6)$$

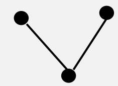
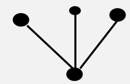
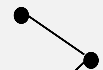
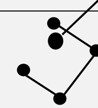
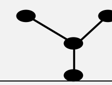
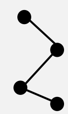
$$b_3 a_{32} c_2^2 + b_4 a_{42} c_2^2 + b_5 a_{52} c_2^2 + b_5 a_{52} c_3^2 + b_5 a_{54} c_4^2 = 1/12 \quad (7)$$

$$b_4 c_2 a_{32} a_{43} + b_5 c_2 a_{32} a_{53} + b_5 c_2 a_{42} a_{54} + b_5 c_3 a_{43} a_{54} = 1/24 \quad (8)$$

The rooted trees and their partial derivatives for the above (8) equations are represented in table 1 below: (Note:

$c_2 = a_{21}, \quad c_3 = a_{31} + a_{32}, c_4 = a_{41} + a_{42} + a_{43}, c_5 = a_{51} + a_{52} + a_{53} + a_{54}$)

Table-1. Rooted trees and derived equations for y derivatives

Equations	Derivatives	R(T)	Tree	T	$\emptyset(t) = \frac{1}{r(t)}$
$b_1 + b_2 + b_3 + b_4 + b_5 = 1$	f	1		T	$\sum_{i=1}^{5} b_i = 1$
$b_2c_2 + b_3c_3 + b_4c_4 + b_5c_5 = 1/2$	ff_y	2		$[t]$	$\sum_{i=2}^{5} b_i c_i = 1/2$
$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 + b_5c_5^2 = 1/3$	ff_y^2	3		$[_2t_2]$	$\sum_{i=2}^{5} b_i c_i^2 = 1/3$
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 = 1/4$	f^3f_{yyy}	4		$[t^3]$	$\sum_{i=2}^{5} b_i c_i^3 = 1/4$
$b_3c_2a_{32} + b_4c_2a_{42} + b_4c_3a_{43} + b_5c_2a_{52} + b_5c_3a_{53} + b_5c_4a_{54} = 1/6$	f^2f_{yy}	3		$[t^2]$	$\sum_{i=3,j=2}^{5,4} b_i a_{ij}c_j = 1/6$
$b_3a_{32}c_2c_3 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 + b_5c_2c_5a_{52} + b_5a_{53}c_3c_5 + b_5c_4c_5a_{54} = 1/8$	$f^2f_yf_{yy}$	4		$[t[t]]$	$\sum_{i=3,j=2}^{5,4} b_i c_i a_{ij}c_j = 1/8$
$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_5a_{52}c_2^2 + b_5a_{52}c_3^2 + b_5a_{54}c_4^2 = 1/12$	$f^2f_yf_{yy}$	4		$[_2t^2]_2$	$\sum_{i=3,j=2}^{5,4} b_i a_{ij}c_j^2 = 1/12$
$b_4c_2a_{32}a_{43} + b_5c_2a_{32}a_{53} + b_5c_2a_{42}a_{54} + b_5c_3a_{43}a_{54} = 1/24$	ff_y^3	4		$[_3t_3]$	$\sum_{i=4,j=3,k=2}^{5,4,3} b_i a_{ij}a_{jk}c_k = 1/24$

Also Putting the $k_{i's}^{i}$ (x, y derivatives only) into $y_{n+1} = y_n + \mathrm{h}(\mathrm{b}_1\mathrm{k}_1 + \mathrm{b}_2\mathrm{k}_2 + \mathrm{b}_3\mathrm{k}_3 + \mathrm{b}_4\mathrm{k}_4 + \mathrm{b}_5\mathrm{k}_5$ where $\phi(x, y, h) = b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5$ and equating coefficients with the Taylor series expansion:

$$\Phi_T(x, y, h) = f + \frac{h}{2!}f_x + \frac{h^2}{3!}\left(f_{xx} + 2ff_{xy} + f_xf_y\right) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + f_xf_y^2)$$

The Equations become:

$$b_1 + b_2 + b_3 + b_4 + b_5 = 1 \quad (9)$$

$$b_2c_2 + b_3c_3 + b_4c_4 + b_5c_5 = {}^1/_2 \quad (10)$$

$$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 + b_5c_5^2 = 1/3 \quad (11)$$

$$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 + b_5c_5^2 = 1/3 \quad (12)$$

$$b_3c_2a_{32} + b_4c_2a_{42} + b_4c_3a_{43} + b_5c_2a_{53} + b_5c_4a_{54} = {}^1/_6 \quad (13)$$

$$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 = 1/4 \quad (14)$$

$$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 = 1/4 \quad (15)$$

$$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 = 1/4 \quad (16)$$

$$b_3a_{32}c_2c_3 + b_4c_2c_4a_{42} + b_4a_{43}c_3c_4 + b_5c_2c_5a_{52} + b_5c_3c_5a_{53} + b_5c_4c_5a_{54} = 1/8 \quad (17)$$

$$b_3a_{32}c_2c_3 + b_4c_2c_4a_{42} + b_4a_{43}c_3c_4 + b_5c_2c_5a_{52} + b_5c_3c_5a_{53} + b_5c_4c_5a_{54} = 1/8 \quad (18)$$

$$b_3a_{32}c_2c_3 + b_4c_2c_4a_{42} + b_4a_{43}c_3c_4 + b_5c_2c_5a_{52} + b_5c_3c_5a_{53} + b_5c_4c_5a_{54} = 1/8 \quad (19)$$

$$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 + b_5a_{52}c_2^2 + b_5a_{53}c_3^2 + b_5a_{54}c_4^2 = {}^1/_{12} \quad (20)$$

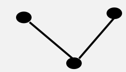
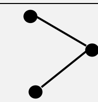
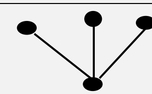
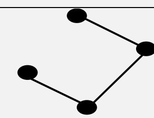
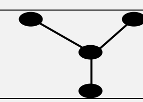
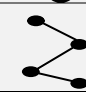
$$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 + b_5a_{52}c_2^2 + b_5a_{53}c_3^2 + b_5a_{54}c_4^2 = {}^1/_{12} \quad (21)$$

$$b_3c_2a_{32}a_{43} + b_5c_2a_{32}a_{53} + b_5c_2a_{42}a_{54} + b_5c_3a_{43}a_{54} = {}^1/_{24} \quad (22)$$

Below is Table 2 Showing the rooted trees for the above fourteen (14) equations (Equations 9-22) and their partial derivatives: (Note:

$$c_2 = a_{21}, \quad c_3 = a_{31} + a_{32}, c_4 = a_{41} + a_{42} + a_{43}, c_5 = a_{51} + a_{52} + a_{53} + a_{54})$$

Table-2. Rooted trees and derived equations for x,y derivatives

Equations	Derivatives	R(T)	TREE	T	$\emptyset(t) = \frac{1}{r(t)}$
$b_1 + b_2 + b_3 + b_4 + b_5 = 1$	f	1		t	$\sum_{i=1}^{5} b_i = 1$
$b_2c_2 + b_3c_3 + b_4c_4 + b_5c_5 = {}^1/_2$	f_x	2		$[t]$	$\sum_{i=2}^{5} b_i\, c_i = {}^1/_2$
$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 + b_5c_5^2 = 1/3$ $b_2c_2^2 + b_3c_3^2 + b_4c_4^2 + b_5c_5^2 = 1/3$	f_{xx} $f f_{xy}$	3 3		$[t^2]$	$\sum_{i=2}^{5} b_i\, c_i^2 = {}^1/_3$
$b_3c_2a_{32} + b_4c_2a_{42} + b_4c_3a_{43} + b_5c_2a_{52} + b_5c_3a_{53} + b_5c_4a_{54} = {}^1/_6$	$f_x f_y$	3		$[_2t]_2$	$\sum_{i=3,j=2}^{5,4} b_i\, a_{ij} c_j = 1/6$
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 = ¼$	f_{xxx}	4		$[t^3]$	$\sum_{i=2}^{5} b_i\, c_i^3 = {}^1/_4$
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 = ¼$	$f f_{xxy}$				
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 = ¼$	$f^2 f_{xyy}$				
$b_3a_{32}c_2c_3 + b_4c_2c_4a_{42} + b_4a_{43}c_3c_4 + b_5c_2c_5a_{52} + b_5c_3c_5a_{53} + b_5c_4c_5a_{54} = 1/8$	$f f_x f_{yy}$	4		$[t[t]]$	$\sum_{i=3,j=2}^{5,4} b_i\, a_{ij} c_i c_j = {}^1/_8$
$b_3a_{32}c_2c_3 + b_4c_2c_4a_{42} + b_4a_{43}c_3c_4 + b_5c_2c_5a_{52} + b_5c_3c_5a_{53} + b_5c_4c_5a_{54} = 1/8$	$f_x f_{xy}$				
$b_3a_{32}c_2c_3 + b_4c_2c_4a_{42} + b_4a_{43}c_3c_4 + b_5c_2c_5a_{52} + b_5c_3c_5a_{53} + b_5c_4c_5a_{54} = 1/8$	$f f_y f_{xy}$				
$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 + b_5a_{52}c_2^2 + b_5a_{53}c_3^2 + b_5a_{54}c_4^2 = {}^1/_{12}$	$f_{xx} f_y$	4		$[_2t^2]_2$	$\sum_{i=3,j=2}^{5,4} b_i\, a_{ij} c_j^2 = {}^1/_{12}$
$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 + b_5a_{52}c_2^2 + b_5a_{53}c_3^2 + b_5a_{54}c_4^2 = {}^1/_{12}$	$f f_y f_{xy}$				
$b_3c_2a_{32}a_{43} + b_5c_2a_{32}a_{53} + b_5c_2a_{42}a_{54} + b_5c_3a_{43}a_{54} = {}^1/_{24}$	$f_x f_y^2$	4		$[_3t]_3$	$\sum_{i=4,j=3,k=2}^{5,4,3} b_i\, a_{ij} a_{jk} c_k = {}^1/_{24}$

Solving the first eight (8) equations,

Set $c_1 = 0, \quad c_2 = ¼, \quad c_3 = ¼, \quad c_4 = ½, \quad c_5 = 1$ (23)

Equation (2) Becomes: $b_2 + b_3 + 2b_4 + 4b_5 = 2$ (24)

Equation (3) Becomes: $3b_2 + 3b_3 + 12b_4 + 48b_5 = 16$ (25)

Equation (5) Becomes: $b_2 + b_3 + 8b_4 + 64b_5 = 16$ (26)

Solving (1), (24), (25) and (26), we have:

$$b_1 = {}^1/_{6,} b_2 = {}^1/_{2,} b_3 = -{}^1/_{2,} b_4 = {}^2/_{3,} b_5 = {}^1/_{6,} \tag{27}$$

Hence,

Equation (4) becomes: $-3a_{32} + 4a_{42} + 4a_{43} + a_{52} + a_{53} + 2a_{54} = 4$ (28)

Equation (6) becomes: $-3a_{32} + 8a_{42} + 8a_{43} + 4a_{52} + 4a_{53} + 8a_{54} = 12$ (29)

Equation (7) becomes: $-3a_{32} + 4a_{42} + 4a_{43} + a_{52} + a_{53} + 4a_{54} = 8$ (30)

Equation (8) becomes: $3a_{32}a_{43} + a_{32}a_{53} + a_{42}a_{54} + a_{43}a_{54} = 1$ (31)

Setting $a_{32} = {}^1/_{2,} a_{42} = {}^1/_{4,} a_{43} = {}^1/_{4,} (28), (29), (30), (31)\ \ become$:

$$2a_{52} + 2a_{53} + 2a_{54} = 2 \tag{32}$$

$$2a_{52} + 2a_{53} + 8a_{54} = 15 \tag{33}$$

$$8a_{52} + 8a_{53} + 16a_{54} = 19 \tag{34}$$

$$a_{53} + a_{54} = 1 \tag{35}$$

Solving (32), (33), (34) and (35) we have:

$$a_{52} = {}^1/_{2,} a_{53} = -1, \qquad a_{54} = 2 \tag{36}$$

Since, $c_2 = a_{21},\ \therefore\ a_{21} = {}^1/_{4,} c_3 = a_{31} + a_{32} = {}^1/_{4,} a_{31} = -{}^1/_{4,} c_4 = a_{41} + a_{42} + a_{43} =$

${}^1/_{2,} a_{41} = 0,$

$$c_5 = a_{51} + a_{52} + a_{53} + a_{54} = 1,\ \therefore\ a_{51} = -{}^1/_2 \tag{37}$$

Putting all the parameters together,

Hence, the fifth-stage fourth-order method becomes:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 3k_2 - 3k_3 + 4k_4 + k_5)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{h}{4}, y_n + \frac{h}{4}k_1\right)$$

$$k_3 = f(x_n + \frac{h}{4}, y_n + \frac{h}{4}(-k_1 + 2k_2))$$

0					
1/4	1/4				
1/4	- 1/4	1/2			
1/2	0	1/4	1/4		
1	-1/2	1/2	-	2	
	1/6	1/2	-1/2	2/3	1/6

$k_4 = f(x_n + \frac{h}{2}, y_n + \frac{h}{4}(k_2 + k_3))$

$$k_5 = f(x_n + h, y_n + \frac{h}{2}(-k_1 + k_2 - 2k_3 + 4k_4))$$

Above, is the fifth-stage fourth-order explicit Runge-Kutta formula and its Butcher's tableau for the parameters.

3. IMPLEMENTATION OF THE FORMULAS AND RESULTS

The formula is implemented on the initial – value problems below with the aid of FORTRAN programming language:

(i) $y^1 = -y, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \frac{1}{e^{xn}}$

(ii) $y^1 = y, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = e^{xn}$

(iii) $y^1 = 1 + y^2, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \tan(x_n + {}^{\pi}/_{4}), h = 0.1$

(iv) $y^1 = y^2, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \frac{1}{1 - x_n}$

4. RESULTS

PROBLEM 1

XN	YN	TSOL	ERROR
.1D+00	0.9048373958333D+00	0.9048374180360D+00	0.2220262629304D-07
.2D+00	0.8187307128984D+00	0.8187307530780D+00	0.4017953347812D-07
.3D+00	0.7408181661478D+00	0.7408182206817D+00	0.5453391760391D-07
.4D+00	0.6703199802432D+00	0.6703200460356D+00	0.6579243794214D-07
.5D+00	0.6065305852983D+00	0.6065306597126D+00	0.7441432348099D-07
.6D+00	0.5488115552946D+00	0.5488116360940D+00	0.8079943614181D-07
.7D+00	0.4965852184960D+00	0.4965853037914D+00	0.8529541101199D-07
.8D+00	0.4493288759132D+00	0.4493289641172D+00	0.8820397545684D-07
.9D+00	0.4065695699541D+00	0.4065696597406D+00	0.8978653853742D-07
.1D+01	0.3678793509023D+00	0.3678794411714D+00	0.9026913183607D-07

PROBLEM 2

XN	YN	TSOL	ERROR
.1D+00	0.1105170937500D+01	0.1105170918076D+01	-.1942435257085D-07
.2D+00	0.1221402801095D+01	0.1221402758160D+01	-.4293445909909D-07
.3D+00	0.1349858878751D+01	0.1349858807576D+01	-.7117487443864D-07
.4D+00	0.1491824802522D+01	0.1491824697641D+01	-.1048805355897D-06
.5D+00	0.1648721415589D+01	0.1648721270700D+01	-.1448886481903D-06

.6D+00	0.1822118992543D+01	0.1822118800391D+01	-.1921520660364D-06
.7D+00	0.2013752955225D+01	0.2013752707470D+01	-.2477543565860D-06
.8D+00	0.2225541241419D+01	0.2225540928492D+01	-.3129267565072D-06
.9D+00	0.2459603500224D+01	0.2459603111157D+01	-.3890672486406D-06
.1D+01	0.2718282306221D+01	0.2718281828459D+01	-.4777620135066D-06

PROBLEM 3

XN	YN	TSOL	ERROR
.1D+00	0.1223138375177D+01	0.1223048934998D+01	-.8944017893153D-04
.2D+00	0.1508791121547D+01	0.1508497718711D+01	-.2934028360511D-03
.3D+00	0.1896566724908D+01	0.1895765223257D+01	-.8015016503180D-03
.4D+00	0.2467218298234D+01	0.2464962911374D+01	-.2255386859622D-02
.5D+00	0.3415663494701D+01	0.3408223718067D+01	-.7439776634312D-02
.6D+00	0.5366223511315D+01	0.5331855866643D+01	-.3436764467238D-01
.7D+00	0.1200178512345D+02	0.1168137680447D+02	-.3204083189739D+00
.8D+00	0.3895792880835D+03	-.6847956583236D+02	-.4580588539158D+03
.9D+00	0.8694330759979D+38	-.8687627875070D+01	-.8694330759979D+38

PROBLEM 4

XN	YN	TSOL	ERROR
.1D+00	0.1111133175011D+01	0.1111111111111D+01	-.2206389955384D-04
.2D+00	0.1250065951128D+01	0.1250000000000D+01	-.6595112836094D-04
.3D+00	0.1428727602736D+01	0.1428571428571D+01	-.1561741644329D-03
.4D+00	0.1667019624523D+01	0.1666666666667D+01	-.3529578558492D-03
.5D+00	0.2000823003079D+01	0.2000000000000D+01	-.8230030794407D-03
.6D+00	0.2502109562425D+01	0.2500000000000D+01	-.2109562424609D-02
.7D+00	0.3339787877491D+01	0.3333333333333D+01	-.6454544157208D-02
.8D+00	0.5027273147390D+01	0.5000000000000D+01	-.2727314738984D-01
.9D+00	0.1021940517339D+02	0.1000000000000D+02	-.2194051733883D+00

4.1. Prove for Stability

Proof:

$$k_1 = \lambda y,$$

$$k_1 = f\left(y_n + \frac{h}{4}k_1\right) = \lambda\left(y_n + \frac{h\lambda y}{4}\right)$$

$$k_1 = \lambda y\left(1 + \frac{\lambda y}{4}\right)$$

$$k_3 = f\left(y_n + \frac{h}{4}(-k_1 + 2k_2)\right) = \lambda(y_n + \frac{h\lambda y}{4} + \frac{2h}{4}\left(\lambda y\left(1 + \frac{\lambda y}{4}\right)\right)$$

$$k_3 = \lambda\left(y_n - \frac{\lambda yh}{4} + \frac{2h\lambda y}{4} + \frac{2h^2\lambda^2 y}{16}\right)$$

$$k_3 = \lambda y\left(1 + \frac{\lambda y}{4} + \frac{\lambda^2 h^2}{8}\right)$$

$$k_4 = f\left(y_n + \frac{h}{4}k_2 + \frac{h}{4}k_3\right) = \lambda\left(y_n + \frac{\lambda yh}{4}\left(1 + \frac{\lambda y}{4}\right) + \frac{\lambda yh}{4}\left(1 + \frac{\lambda y}{4} + \frac{\lambda^2 h^2}{8}\right)\right)$$

$$k_4 = \lambda y\left(1 + \frac{\lambda y}{4} + \frac{\lambda^2 h^2}{16} + \frac{\lambda y}{4} + \frac{\lambda^2 h^2}{16} + \frac{\lambda^3 h^3}{32}\right)$$

$$k_4 = \lambda y\left(1 + \frac{\lambda y}{2} + \frac{\lambda^2 h^2}{8} + \frac{\lambda^3 h^3}{32}\right)$$

$$k_5 = f\left(y_n - \frac{hk_1}{2} + \frac{hk_2}{2} - \frac{-2hk_3}{2} + \frac{4hk_4}{4}\right)$$

$$k_5 = \lambda\left(y_n - \frac{\lambda yh}{2} + \frac{\lambda yh}{2}\left(1 + \frac{\lambda y}{4}\right) - \frac{2\lambda yh}{4}\left(1 + \frac{\lambda y}{4} + \frac{\lambda^2 h^2}{8}\right)\right.$$
$$\left. + \frac{4\lambda yh}{2}\left(1 + \frac{\lambda y}{2} + \frac{\lambda^2 h^2}{8} + \frac{\lambda^3 h^3}{32}\right)\right)$$

$$k_5 = \lambda y\left(1 - \frac{\lambda y}{2} + \frac{\lambda y}{2} + \frac{\lambda^2 h^2}{8} - \lambda y - \frac{\lambda^2 h^2}{4} - \frac{\lambda^3 h^3}{8} + 2\lambda y + \lambda^2 h^2 + \frac{\lambda^3 h^3}{4}\right)$$

$$k_5 = \lambda y\left(1 + \lambda y + \frac{7}{4}\lambda^2 h^2 + \frac{\lambda^3 h^3}{8}\right)$$

$$y_{n+1} - y_n = \frac{h}{6}\left[\lambda y + 3\lambda y\left(1 + \frac{\lambda y}{4} - 3\lambda y\left(1 + \frac{\lambda y}{4} + \frac{\lambda^2 h^2}{8}\right)\right) + 4\lambda y\left(1 + \frac{\lambda y}{2} + \frac{\lambda^2 h^2}{8} + \frac{\lambda^3 h^3}{32}\right)\right.$$
$$\left. + \lambda y\left(1 + \lambda y + \frac{7}{8}\lambda^2 h^2 + \frac{\lambda^3 h^3}{8}\right)\right]$$

$$y_{n+1} - y_n = \frac{\lambda yh}{6}\left[1 + 3 + \frac{3\lambda y}{4} - 3 - \frac{3\lambda y}{4} - \frac{3\lambda^2 h^2}{8} + 4 + 2\lambda y + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{8} + 1 + \lambda h\right.$$
$$\left. + \frac{7\lambda^2 h^2}{8} + \frac{\lambda^3 h^3}{8}\right]$$

$$y_{n+1} - y_n = \frac{\lambda yh}{6}\left[6 + 3\lambda h + \lambda^2 h^2 + \frac{\lambda^3 h^3}{4}\right]$$

Dividing by y and setting $\mu = \lambda h$, we have:

$$\frac{y_{n+1} - y_n}{y_n} = \frac{\mu}{6}\left[6 + 3\lambda h + \lambda^2 h^2 + \frac{\lambda^3 h^3}{4}\right]$$

$$\frac{y_{n+1}}{y_n} - 1\left[\mu + \frac{\mu^2}{2} + \frac{\mu^3}{6} + \frac{\mu^4}{24}\right]$$

$$\frac{y_{n+1}}{y_n} = 1 + \mu + \frac{\mu^2}{2} + \frac{\mu^3}{6} + \frac{\mu^4}{24} = 0$$

$$\mu = \frac{-589}{2177}, \frac{-1573}{625} i, \frac{-652}{377}, \frac{-1153}{1297} i,$$

The Stability region is seen in the diagram below plotted using matlab.

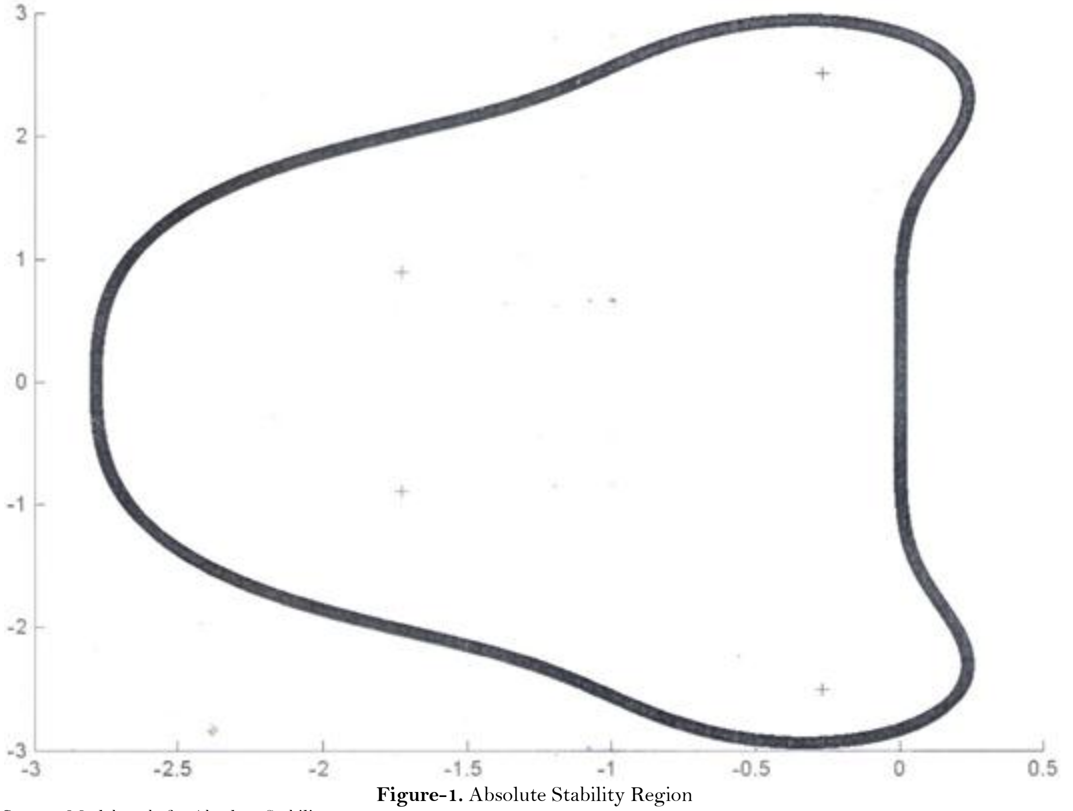


Figure-1. Absolute Stability Region

Source: Matlab code for Absolute Stability

It is clearly seen from the diagram that our method is absolately stable.

4.2. Prove For Convergence and Consistency of the Method

Proof:

$$y_{n-1} - y_n = \frac{h}{6}(k_1 + 3k_2 - 3k_3 + 4k_4 + k_5)$$

$$K_1 = f(x_n, y_n), \quad k_2 = f(x_n + c_2 h, y_n + h a_{21} k_1)$$

$$K_3 = f\big(x_n + c_3 h,\; y_n + h\,(a_{31}k_1 + a_{32}k_2)\big),\; k_4 = f\big(x_n + c_4 h,\; y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\big)$$

$$K_5 = f\big(x_n + c_5 h,\; y_n + h\,(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)\big).$$

The above is convergent and consistent to a known function if $y' = f(x,y) \quad y(a) = \mu, a \le x \le b$.

$$i.e. \emptyset\,(x, y, 0) = f(x, y).$$

$$T_n(h^5) = y_{n-1} - y_n$$

$$= \frac{h}{6}\left(f(x_n, y_n) + 3\left[f\left(x_n + c_2h, y_n + ha_{21}k_1 f(x_n, y_n)\right)\right]\right.$$

$$- 3\left[f\left(x_n + c_3h, \quad y_n\right.\right.$$

$$\left.\left. + h\left(a_{31}f(x_n, y_n) + a_{32}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)\right)\right)\right]$$

$$+ 4\left[f\left(x_n + c_4h, \; y_n\right.\right.$$

$$+ h\left(a_{41}f(x_n, y_n)\right)$$

$$+ a_{42}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)$$

$$+ a_{43}\left(f(x_n + c_3h, y_n + h(a_{31}f(x_n, y_n)\right)$$

$$\left.\left. + a_{32}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)\right)\right)\right)\right)\Bigg] + f(x_n + c_5h, \; y_n$$

$$+ h(a_{51}f(x_n, y_n) + a_{52}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)$$

$$+ a_{53}(f\left(x_n + c_3h, \qquad y_n + h(a_{31}f(x_n, y_n)\right)$$

$$+ a_{32}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)$$

$$+ a_{54}\left(f\left(x_n + c_4h, y_n\right.\right.$$

$$+ h\left(a_{41}f(x_n, y_n) + a_{42}\left(f\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)\right.$$

$$+ a_{43}\left(f\left(x_n + c_3h,\right.\right.$$

$$\left.\left.\left.\left. y_n + h(a_{31}f(x_n, y_n) + a_{32}\left(f\left(\left(x_n + c_2h, y_n + ha_{21}f(x_n, y_n)\right)\right)\right)\right)\right)\right)\right)\right)\Bigg]$$

$Dividing\ all\ through\ by\ h\ and\ taking\ the\ limit\ of\ both\ side\ as\ h \to 0, we\ have$

$$h_n(h) = \frac{y_{n-1} - y_n}{h} = \frac{1}{6}\left[f(x_n, y_n) + 3f(x_n, y_n) - 3f(x_n, y_n) + 4f(x_n, y_n) + f(x_n, y_n)\right]$$

$$= \frac{1}{6}[6f(x_n, y_n)] = f(x_n, y_n)$$

$$= \emptyset(x_n, y_n, o) = f(x_n, y_n), y(x_0) = y_0.$$

Hence our method is convergent and consistent.

5. FINDINGS, CONTRIBUTION TO KNOWLEDGE AND CONCLUSION

This study contributes in the existing work of Butcher in [18, 19, 22, 23]. The study uses new estimation methodology to show that the two sets of equations derived from both derivatives (y derivatives and x,y derivatives) are the same, hence generating the same rooted trees. The paper's primary contribution is to find easy approach in deriving a new Runge-Kutta formula. This study originates a new formula that is absolutely stable and consistent in handling problems in Ordinary Differential Equations.

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APPENDIX

FORTRAN PROGRAM THAT GENERATED THE RESULTS

```
C PRO
C SECOND-ORDER CLASSICAL RUNGE-KUTTA METHOD
C OUR PROBLEM IS :Y'=-Y,Y(0)=1
C THEORETICAL SOLUTION:Y(XN)=EXP(XN)
      DOUBLE PRECISION XN,YN,H,ONE,TWO,FOUR,SIX,THREE,TWELVE
      DOUBLE PRECISION TTWO,FSIX,TTHREE,ONONE,SFOUR
      DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,K5,EIGHT
OPEN(6,FILE='RUNG2.OUT')
      H=0.1D0
      YN=1.0D0
      XN=0.1D0
      THREE=3.0D0
      ONE=1.0D0
      TWO=2.0D0
      FOUR=4.0D0
      TTWO=32.0D0
      FSIX=46.0D0
      TTHREE=23.0D0
      ONONE=111.0D0
      SFOUR=74.0D0
```

```
      SIX=6.0D0
      TWELVE=12.0D0
      EIGHT=8.0D0
      ONT=192.0D0
      PI=FOUR*DATAN(ONE)
WRITE(6,101)
 3   K1=(YN)
      K2=(YN+(H/FOUR)*K1)
      K3=(YN+(H/FOUR)*(-K1+TWO*K2))
      K4=(YN+(H/FOUR)*(K2+K3))
      K5=(YN+(H/TWO)*(-K1+K2-TWO*K3+FOUR*K4))
      YN=YN+(H/SIX)*(K1+THREE*K2-THREE*K3+FOUR*K4+K5)
      TSOL=EXP(XN)
      ERROR=TSOL-YN
WRITE(6,100)XN,YN,TSOL,ERROR
      XN=XN+H
IF(XN.LE.ONE) GOTO 3
 100   FORMAT(D6.1,1X,3(3X,D19.13))
 101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
      END
C PRO
C SECOND-ORDER CLASSICAL RUNGE-KUTTA METHOD
C OUR PROBLEM IS :Y'=-Y,Y(0)=1
C THEORETICAL SOLUTION:Y(XN)=EXP(XN)
      DOUBLE PRECISION XN,YN,H,ONE,TWO,FOUR,SIX,THREE
      DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,K5
OPEN(6,FILE='RUNG2.OUT')
      H=0.1D0
      YN=1.0D0
      XN=0.1D0
      THREE=3.0D0
      ONE=1.0D0
      TWO=2.0D0
      FOUR=4.0D0
      SIX=6.0D0
      EIGHT=8.0D0
      PI=FOUR*DATAN(ONE)
WRITE(6,101)
```

```
  3    K1=-(YN)
       K2=-(YN+(H/FOUR)*K1)
       K3=-(YN+(H/FOUR)*(-K1+TWO*K2))
       K4=-(YN+(H/FOUR)*(K2+K3))
       K5=-(YN+(H/TWO)*(-K1+K2-TWO*K3+FOUR*K4))
       YN=YN+(H/SIX)*(K1+THREE*K2-THREE*K3+FOUR*K4+K5)
       TSOL=ONE/EXP(XN)
       ERROR=TSOL-YN
WRITE(6,100)XN,YN,TSOL,ERROR
       XN=XN+H
IF(XN.LE.ONE) GOTO 3
 100   FORMAT(D6.1,1X,3(3X,D19.13))
 101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
       END
C PRO
C SECOND-ORDER CLASSICAL RUNGE-KUTTA METHOD
C OUR PROBLEM IS :Y'=-Y,Y(0)=1
C THEORETICAL SOLUTION:Y(XN)=EXP(XN)
       DOUBLE PRECISION XN,YN,H,ONE,TWO,FOUR,SIX,THREE
       DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,K5
OPEN(6,FILE='RUNG2.OUT')
       H=0.1D0
       YN=1.0D0
       XN=0.1D0
       THREE=3.0D0
       ONE=1.0D0
       TWO=2.0D0
       FOUR=4.0D0
       SIX=6.0D0
       EIGHT=8.0D0
       PI=FOUR*DATAN(ONE)
WRITE(6,101)
  3    K1=(YN)**TWO
       K2=(YN+(H/FOUR)*K1)**TWO
       K3=(YN+(H/FOUR)*(-K1+TWO*K2))**TWO
       K4=(YN+(H/FOUR)*(K2+K3))**TWO
       K5=(YN+(H/TWO)*(-K1+K2-TWO*K3+FOUR*K4))**TWO
       YN=YN+(H/SIX)*(K1+THREE*K2-THREE*K3+FOUR*K4+K5)
```

```
      TSOL=ONE/(ONE-XN)
      ERROR=TSOL-YN
WRITE(6,100)XN,YN,TSOL,ERROR
      XN=XN+H
IF(XN.LE.ONE) GOTO 3
 100   FORMAT(D6.1,1X,3(3X,D19.13))
 101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
      END
C PRO
C SECOND-ORDER CLASSICAL RUNGE-KUTTA METHOD
C OUR PROBLEM IS :Y'=-Y,Y(0)=1
C THEORETICAL SOLUTION:Y(XN)=EXP(XN)
      DOUBLE PRECISION XN,YN,H,ONE,TWO,FOUR,SIX,THREE
      DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,K5
OPEN(6,FILE='RUNG2.OUT')
      H=0.1D0
      YN=1.0D0
      XN=0.1D0
      THREE=3.0D0
      ONE=1.0D0
      TWO=2.0D0
      FOUR=4.0D0
      SIX=6.0D0
      EIGHT=8.0D0
      PI=FOUR*DATAN(ONE)
WRITE(6,101)
  3    K1=(YN)**TWO+ONE
      K2=(YN+(H/FOUR)*K1)**TWO+ONE
      K3=(YN+(H/FOUR)*(-K1+TWO*K2))**TWO+ONE
      K4=(YN+(H/FOUR)*(K2+K3))**TWO+ONE
      K5=(YN+(H/TWO)*(-K1+K2-TWO*K3+FOUR*K4))**TWO+ONE
      YN=YN+(H/SIX)*(K1+THREE*K2-THREE*K3+FOUR*K4+K5)
      TSOL=TAN(XN+(PI/FOUR))
      ERROR=TSOL-YN
WRITE (6,100)XN,YN,TSOL,ERROR
      XN=XN+H
IF (XN.LE.ONE) GOTO 3
 100   FORMAT (D6.1,1X,3(3X,D19.13))
```

```
101  FORMAT (2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
     END
```

MATLAB CODE FOR PLOTTING THE REGION OF ABSOLUTE STABILITY

```
Q = 0:0.001:2*pi
a = zeros (4,length(Q))
For k = 1: length(Q)
c = [1/24 1/6 1/2 1 1-exp (i*Q(k))]
a (:,k)=roots(c)
b=roots(c)
End
Hold on
plot (a(1,:), 'ko')
Plot (a(2,:), 'ko')
Plot (a(3,:), 'ko')
plot (a(4,:), 'ko')
Hold off               `
```

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