



RICCATI'S EQUATION. ASYMPTOTICS OF EXACT SOLUTION

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ABSTRACT

In this article based on a method of approximating equation an asymptotic solution of the general Riccati's equation is obtained. The principal distinctive feature and advantage of the solution is its continuity at turning points. Estimates of accuracy of the approximate solution are derived. Limit values of the asymptotic solution in case of one-sided convergence of argument to turning point of the first order are calculated.

Keywords: General Riccati's equation, Asymptotic solutions, Turning points, Approximate equations, Asymptotic calculation of integral, small parameter

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Contribution/ Originality

This study contributes in the existing literature. Thus, the results of that study may be applied for solving problems were set up in works [1], [2], [3]. This study is one of very few studies, which have investigated the asymptotic behavior of solutions of Riccati's equation in the neighborhood of turning points.

1. PRINCIPAL ASYMPTOTIC EXPANSION

We examine Riccati equation with assigned small positive parameter ε :

$$q' = i \left(\frac{r(x)}{\varepsilon^2} - q^2 \right). \quad (1)$$

Let us rewrite this equation in form of

$$\frac{\varepsilon}{\sqrt{r}} \frac{q'}{q} = i \left(\frac{\sqrt{r}}{\varepsilon} \frac{1}{q} - \frac{\varepsilon}{\sqrt{r}} q \right), \quad (2)$$

Where we will understand the root $\sqrt{r(x)}$ as one of its two branches. Note that in domain where $r(x) \neq 0$ the asymptotic solution with $\varepsilon \rightarrow 0$ is function $\frac{\sqrt{r(x)}}{\varepsilon}$. Thus, asymptotic expansion of exact solution by parameter ε powers should be constructed as

$$q = \frac{\sqrt{r}}{\varepsilon} + \sum_{n=0}^{\infty} q_n \varepsilon^n. \tag{3}$$

This series' coefficients are defined by its substitution in equation (1) and by equating of terms with same ε powers. Using the identity

$$\left(\sum_{n=0}^{\infty} q_n \varepsilon^n \right)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n q_k q_{n-k} \right) \varepsilon^n,$$

We come to recurrence equation

$$q_0 = -\frac{1}{4i} \frac{r'}{r}, \tag{4}$$

$$q_{n+1} = \frac{1}{2\sqrt{r}} \left(iq'_n - \sum_{k=0}^n q_k q_{n-k} \right). \tag{5}$$

Make attention that in order to calculate coefficient q_n we have to assume $r(x)$ function $n+1$ -multiple differentiability. The asymptotic solution with accuracy to order ε looks like this

$$q = \frac{\sqrt{r}}{\varepsilon} - \frac{1}{4i} \frac{r'}{r} + \frac{1}{8\sqrt{r}} \left(\frac{5}{4} \left(\frac{r'}{r} \right)^2 - \frac{r''}{r} \right) \varepsilon + O(\varepsilon^2). \tag{6}$$

2. APPROXIMATE EQUATIONS

Let us introduce the designation

$$k(x) = \frac{\sqrt{r(x)}}{\varepsilon}$$

Simplifying considerably the record of many subsequent expressions.

Now equation (2) may be represented as

$$\frac{q'}{2ikq} = \text{sh} \left(\ln \frac{k}{q} \right). \tag{7}$$

As $q \rightarrow k(x)$ with $\varepsilon \rightarrow 0$, then following approximate equality is true

$$\operatorname{sh}\left(\ln \frac{k}{q}\right) \approx \ln \frac{k}{q}.$$

Having replaced the right side of the equation (7) accordingly to this equality, we will get following approximate equation:

$$\frac{q'}{2ikq} = \ln \frac{k}{q}. \tag{8}$$

This is linear first order differential equation concerning $\ln q$. Thus, it admits explicit form of solution. For the first time it was treated in the work [4]. Let us write out this solution assuming $q(x_0) = k(x_0) = k_0$. To discriminate it from Riccati equation exact solution we will designate it by capital Q .

$$Q = k_0 \exp \left[2ie^{-2i \int k dx} \int_{x_0}^x k \ln \frac{k}{k_0} e^{2i \int k dx} dx \right]. \tag{9}$$

When substituting in that solution ε for $-\varepsilon$ we will obtain other asymptotic solution of equation (1) for which with $\varepsilon \rightarrow 0$ the main term of asymptotic expansion will be function

$$-\frac{\sqrt{r}}{\varepsilon}.$$

3. APPROXIMATE SOLUTION ACCURACY EVALUATIONS

We will evaluate obtained approximate solution by comparing its asymptotic and power expansions to corresponding expansions of Riccati equation exact solution.

We will start from asymptotic expansions. Remembering that $k = \frac{\sqrt{r}}{\varepsilon}$, we are looking for

solution of equation (8) in form of asymptotic series by parameter ε powers. As a result of necessary operations for function (9), we obtain such asymptotic formula

$$Q = \frac{\sqrt{r}}{\varepsilon} - \frac{1}{4i} \frac{r'}{r} + \frac{1}{8\sqrt{r}} \left(\frac{5}{4} \left(\frac{r'}{r} \right)^2 - \frac{r''}{r} \right) \varepsilon + O(\varepsilon^2). \tag{10}$$

Comparing this formula to the formula (6), we can see that $|q - Q| \sim O(\varepsilon^2)$.

Let us examine now power series of argument x . Differential equations solutions expansion in x degrees is obtained by their successive differentiation and calculation of values of derivatives in one fixed initial point. For Riccati equation (1) and approximate equation (8), this process relates to very bulky calculations. It will become little simpler if we assume that in the initial point $x_0 = 0$ conditions are met that $r(0) = r_0 > 0, r'(0) = 0$. In this case, for Riccati equation (1) mentioned calculations result in following values:

$$q(0) = \frac{r_0^{1/2}}{\varepsilon}, \quad q'(0) = 0, \quad q''(0) = 0, \quad q'''(0) = \frac{ir''(0)}{\varepsilon^2},$$

$$q^{(4)}(0) = \frac{i}{\varepsilon^2} r'''(0) + \frac{2}{\varepsilon^3} r_0^{1/2} r''(0),$$

$$q^{(5)}(0) = \frac{i}{\varepsilon^2} r^{(4)}(0) + \frac{2}{\varepsilon^3} r_0^{1/2} r'''(0) - \frac{4i}{\varepsilon^4} r_0 r''(0),$$

$$q^{(6)}(0) = \frac{i}{\varepsilon^2} r^{(5)}(0) + \frac{2}{\varepsilon^3} r_0^{1/2} r^{(4)}(0) - \frac{4i}{\varepsilon^4} r_0 r'''(0) - \frac{8}{\varepsilon^5} r_0^{3/2} r''(0),$$

$$q^{(7)}(0) = \frac{i}{\varepsilon^2} r^{(6)}(0) + \frac{2}{\varepsilon^3} r_0^{1/2} r^{(5)}(0) - \frac{4i}{\varepsilon^4} (r_0 r^{(4)}(0) - 5r''^2(0)) - \frac{8}{\varepsilon^5} r_0^{3/2} r'''(0) + \frac{16i}{\varepsilon^6} r_0^2 r''(0). \quad \square$$

Making similar calculations to solve of equation (8) and comparing them to values of derivatives that have just been written we will get power corrections for approximate solution deflection from Riccati equation exact solution. For function (9) the correction will be as follows

$$q = Q + \frac{i}{448\varepsilon^2} \frac{r''^3(0)}{r_0^2} x^7 + O(x^8). \quad (11)$$

4. RICCATI EQUATION REAL-VALUED FORMS AND THEIR APPROXIMATE SOLUTIONS

In the domain where $r(x) < 0$, it is more practical to use Riccati equation (1) in two real-valued forms. These forms come out by means of having introduced designation

$$q = i\gamma_1 \text{ or } q = -i\gamma_2, \quad \chi(x) = \frac{\sqrt{-r(x)}}{\varepsilon}.$$

Hence

$$\gamma_1' = \gamma_1^2 - \chi^2, \quad \gamma_2' = \chi^2 - \gamma_2^2. \tag{12}$$

Approximate solutions of these equations are deduced absolutely similarly to deduction of approximate solution for Riccati equation complex-valued form (1), or can be simply obtained from formula (9) by due substitutions. Let us write down these solutions admitting conditions

$$\gamma_1(x_1) = \chi(x_1) = \chi_1, \quad \gamma_2(x_2) = \chi(x_2) = \chi_2.$$

$$\gamma_1 = \chi_1 \exp \left[-2e^{2\int \chi dx} \int_{x_1}^x \chi \ln \frac{\chi}{\chi_1} e^{-2\int \chi dx} dx \right], \tag{13}$$

$$\gamma_2 = \chi_2 \exp \left[2e^{-2\int \chi dx} \int_{x_2}^x \chi \ln \frac{\chi}{\chi_2} e^{2\int \chi dx} dx \right]. \tag{14}$$

Note that at $\varepsilon \rightarrow 0$ both these approximate solutions tend asymptotically to the same function $\chi(x)$.

5. ASYMPTOTIC CALCULATION OF INTEGRALS. LIMIT VALUES OF APPROXIMATE SOLUTIONS

One of the most important issues in Riccati equation theory is the one of solution value calculation in zero of coefficient $r(x)$. Point $x=a$ where $r(a)=0$, is called turning point. Note that all approximate solutions obtained in previous paragraphs remain finite in turning points. However, above-mentioned evaluations do not allow judging how they reflect exact solutions' behavior in neighborhood of these points. Maybe analysis of approximate solution (9) as complex variable functions will clarify the situation. The author leaves this problem open. Nevertheless,

concrete examples show that using limit values of function $Q(x), \gamma_1(x), \gamma_2(x)$ assuming unilateral tendency of argument x to a turning point, leads to correct results!

Further relevant calculations are given and principal terms of asymptotic values in turning points are obtained.

Let point $x = 0$ be a turning point, i.e. $r(0) = 0$. It is necessary to examine separately four situations represented in figures 1, 2, 3, 4.

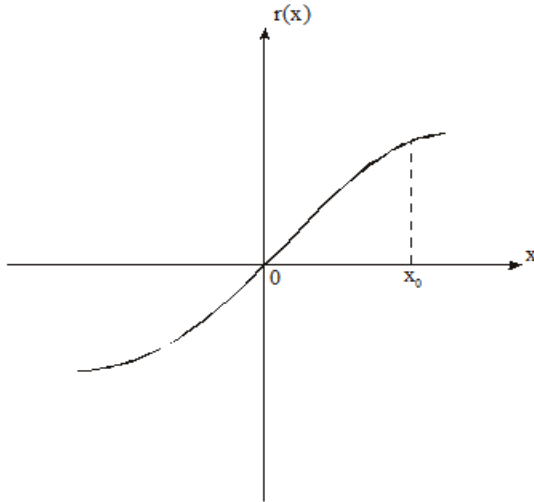


Fig-1. $r'(0) > 0$. Argument x tends to the turning point from the right.
Source: See [5]

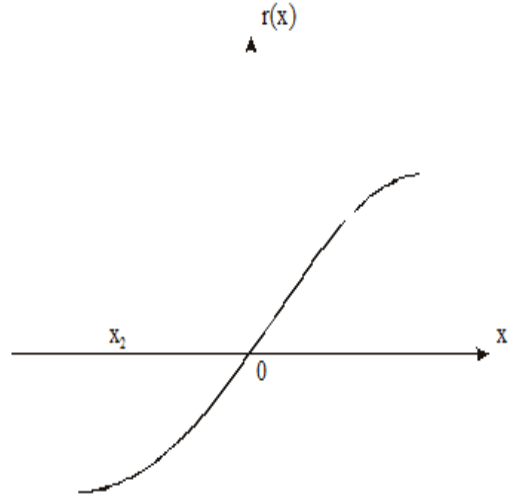


Fig-2. $r'(0) > 0$. Argument x tends to the turning point from the left.
Source: See [5]

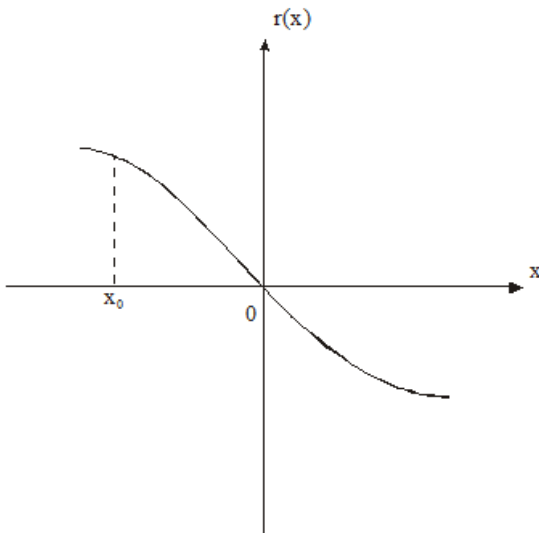


Fig-3. $r'(0) < 0$. Argument x tends to the turning point from the left.
Source: See [5]

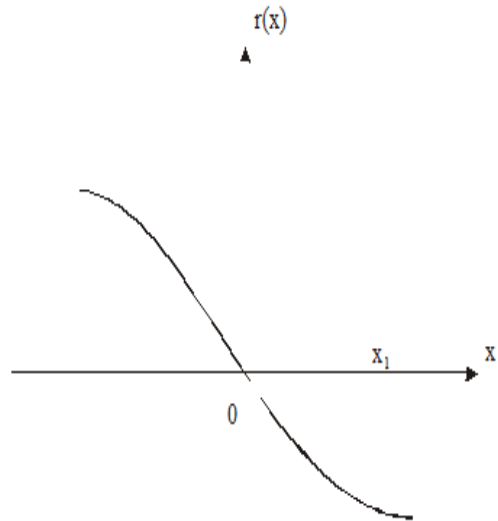


Fig-4. $r'(0) < 0$. Argument x tends to the turning point from the right.
Source: See [5]

We assume that in neighborhood of point $x = 0$ function $r(x)$ is analytic, i.e. representable by a power series

$$r(x) = r_1x + r_2x^2 + r_3x^3 + \dots \quad , \quad (15)$$

Where coefficient $r_1 = r'(0)$ is positive in cases corresponding to figures 1, 2 and negative in cases corresponding to figures 3, 4.

Let us put

$$s(x) = \int_0^x \sqrt{r(x)} dx, \quad \text{at } r(x) > 0, \tag{16}$$

and

$$s(x) = \int_0^x \sqrt{-r(x)} dx, \quad \text{at } r(x) < 0.$$

Then in both these situations we have

$$\sqrt{|r(x)|} = \sqrt{|r_1 x|} \left(1 + \frac{1}{2} \frac{r_2}{r_1} x + \dots \right), \tag{17}$$

$$|s(x)| = \frac{2}{3} \sqrt{|r_1 x^3|} \left(1 + \frac{3}{10} \frac{r_2}{r_1} x + \dots \right), \tag{18}$$

$$h(x) = \delta \frac{\sqrt{|r(x)|}}{\sqrt[3]{|s(x)|}}, \tag{19}$$

where

$$\delta = \sqrt[3]{\frac{2}{3|r_1|}},$$

$$h(x) = \frac{1 + \frac{1}{2} \frac{r_2}{r_1} x + \dots}{\left(1 + \frac{3}{10} \frac{r_2}{r_1} x + \dots \right)^{\frac{1}{3}}} = 1 + \frac{2}{5} \frac{r_2}{r_1} x + \dots .$$

Using these correlations, we will mark out dominant terms in asymptotic of values of $Q(0)$, $\gamma_1(0)$, $\gamma_2(0)$ calculated by formulae (9) and (13), (14).

1. Situation corresponding to figure 1. $r_1 > 0$, $x_0 > 0$. We calculate in point $x = 0$ dominant term of asymptotic of integral taken from formula (9) and then using it, we find value of $Q(0)$.

$$\begin{aligned} 2i \int_0^{x_0} k \ln \frac{k}{k_0} e^{\frac{2i}{\varepsilon} \int_0^x k dx} dx &= \int_0^{x_0} \ln \frac{k}{k_0} \cdot d \left(e^{\frac{2i}{\varepsilon} \int_0^x k dx} - 1 \right) = \\ &= \left(e^{\frac{2i}{\varepsilon} \int_0^x k dx} - 1 \right) \ln \frac{k}{k_0} \Big|_0^{x_0} - \int_0^{x_0} \left(e^{\frac{2i}{\varepsilon} \int_0^x k dx} - 1 \right) d \ln \frac{k}{k_0} = \\ &= - \int_0^{x_0} \left(e^{\frac{2i}{\varepsilon} \int_0^x \sqrt{r} dx} - 1 \right) d \ln \sqrt{r} = \end{aligned}$$

in this case $\sqrt{r} = \sqrt{|r|}$, $|s| = s$, and correlation (19) gives $\sqrt{r} = \delta^{-1} s^{\frac{1}{3}} h$, so:

$$\begin{aligned} &= - \int_0^{x_0} \left(e^{\frac{2i}{\varepsilon} s(x)} - 1 \right) \cdot d \ln \left(\delta^{-1} s(x)^{\frac{1}{3}} h(x) \right) = \\ &= - \frac{1}{3} \int_0^{s(x)} \frac{e^{\frac{2i}{\varepsilon} s} - 1}{s} ds - \int_0^{x_0} \left(e^{\frac{2i}{\varepsilon} s(x)} - 1 \right) \frac{h'(x)}{h(x)} dx = \left. \begin{array}{l} t = \frac{2}{\varepsilon} s \\ s = \frac{\varepsilon}{2} t \\ ds = \frac{\varepsilon}{2} dt \end{array} \right| = \\ &= - \frac{1}{3} \int_0^{\frac{2}{\varepsilon} s(x_0)} \frac{e^{it} - 1}{t} dt - \int_0^{x_0} \frac{h'(x)}{h(x)} e^{\frac{2i}{\varepsilon} s(x)} dx + \int_0^{x_0} \frac{h'(x)}{h(x)} dx = \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{3} \int_0^{\frac{2}{\varepsilon}s(x_0)} \frac{\cos t - 1}{t} dt - \frac{1}{3} i \int_0^{\frac{2}{\varepsilon}s(x_0)} \frac{\sin t}{t} dt + \ln h(x_0) - \int_0^{x_0} \frac{h'(x)}{h(x)} e^{\frac{2i}{\varepsilon}s(x)} dx = \\
 &= -\frac{1}{3} \left(-\int_{\frac{2}{\varepsilon}s(x_0)}^{\infty} \frac{\cos t}{t} dt - \ln \frac{2}{\varepsilon}s(x_0) - C \right) - \frac{1}{3} i \left(-\int_{\frac{2}{\varepsilon}s(x_0)}^{\infty} \frac{\sin t}{t} dt + \frac{\pi}{2} \right) + \ln h(x_0) - \int_0^{x_0} \frac{h'(x)}{h(x)} e^{\frac{2i}{\varepsilon}s(x)} dx = \\
 &= \ln \left(\sqrt[3]{\frac{2}{\varepsilon}s^3(x_0)} h(x_0) \right) + \frac{C}{3} - i \frac{\pi}{6} + O(\varepsilon) = \\
 &= \ln \left(\sqrt[3]{\frac{2}{\varepsilon} \delta \sqrt{r(x_0)}} \right) + \frac{C}{3} - i \frac{\pi}{6} + O(\varepsilon), \tag{20}
 \end{aligned}$$

where

$C = 0,577216$ is Euler constant,

$$O(\varepsilon) = \frac{1}{3} \int_{\frac{2}{\varepsilon}s(x_0)}^{\infty} \frac{e^{it}}{t} dt - \int_0^{x_0} \frac{h'(x)}{h(x)} e^{\frac{2i}{\varepsilon}s(x)} dx.$$

Value of $Q(0)$ is calculated by formula (9) as follows

$$Q(0) = k_0 \exp \left[-2i \int_0^{x_0} k \ln \frac{k}{k_0} e^{2i \int_0^x k dx} dx \right].$$

Introducing in this dominant term of asymptotic of integral (20), we get

$$Q(0) = \frac{1}{\varepsilon^{\frac{2}{3}}} \sqrt[3]{\frac{3r_1}{4}} e^{-\frac{C}{3} + i\frac{\pi}{6}} (1 - O(\varepsilon)). \tag{21}$$

2. Situation corresponding to figure 2, $r_1 > 0$, $x_2 < 0$. We calculate in point $x = 0$ dominant term of asymptotic of integral taken from formulae (14) and then using them we find value $\gamma_2(0)$.

$$\begin{aligned}
 & 2 \int_{x_2}^0 \chi \ln \frac{\chi}{\chi_2} e^{2 \int_0^x \chi dx} dx = \int_{x_2}^0 \ln \frac{\chi}{\chi_2} d \left(e^{2 \int_0^x \chi dx} - 1 \right) = \\
 & = \left(e^{2 \int_0^x \chi dx} - 1 \right) \ln \frac{\chi}{\chi_2} \Big|_{x_2}^0 - \int_{x_2}^0 \left(e^{2 \int_0^x \chi dx} - 1 \right) d \ln \frac{\chi}{\chi_2} = \\
 & = - \int_{x_2}^0 \left(e^{\frac{2}{\varepsilon} \int_0^x \sqrt{-r} dx} - 1 \right) d \ln \sqrt{-r} =
 \end{aligned}$$

In this case $\sqrt{-r} = \sqrt{|r|}$, $s = -|s|$ ($x < 0$) and correlation (19) gives

$$\sqrt{-r} = -\delta^{-1} s^{\frac{1}{3}} h, \text{ so:}$$

$$\begin{aligned}
 & = \int_{x_2}^0 \left(e^{\frac{2}{\varepsilon} s} - 1 \right) d \ln \left(-\delta^{-1} s^{\frac{1}{3}} h \right) = \\
 & = -\frac{1}{3} \int_{s(x_2)}^0 \frac{e^{\frac{2}{\varepsilon} s} - 1}{s} ds - \int_{x_2}^0 \left(e^{\frac{2}{\varepsilon} s} - 1 \right) \frac{h'}{h} dx = \left. \begin{array}{l} t = \frac{2}{\varepsilon} |s| = -\frac{2}{3} s \\ s = -\frac{\varepsilon}{2} t \\ ds = -\frac{\varepsilon}{2} dt \end{array} \right| = \\
 & = -\frac{1}{3} \int_{\frac{2}{\varepsilon} |s(x_2)|}^0 \frac{e^{-t} - 1}{t} dt + \int_{x_2}^0 \frac{h'}{h} dx - \int_{x_2}^0 \frac{h'}{h} e^{\frac{2}{\varepsilon} s} dx = \\
 & = \frac{1}{3} \int_0^{\frac{2}{\varepsilon} |s(x_2)|} \frac{e^{-t} - 1}{t} dt - \ln h(x_0) - \int_{x_2}^0 \frac{h'}{h} e^{\frac{2}{\varepsilon} s} dx =
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{3} \int_{\frac{2}{\varepsilon} |s(x_2)|}^{\infty} \frac{e^{-t}}{t} dt - \frac{1}{3} \ln \frac{2}{\varepsilon} |s(x_2)| - \frac{1}{3} C - \ln h(x_2) - \int_{x_2}^0 \frac{h'}{h} e^{\frac{2}{\varepsilon} s} dx = \\
 &= -\ln \left(\sqrt[3]{\frac{2}{\varepsilon} |s(x_2)|} h(x_0) \right) - \frac{C}{3} + O(\varepsilon) = \\
 &= -\ln \left(\sqrt[3]{\frac{2}{\varepsilon} \delta \sqrt{-r(x_2)}} \right) - \frac{C}{3} + O(\varepsilon), \tag{22}
 \end{aligned}$$

where C is Euler constant,

$$O(\varepsilon) = -\frac{1}{3} \int_{\frac{2}{\varepsilon} |s(x_2)|}^{\infty} \frac{e^{-t}}{t} dt - \int_{x_2}^0 \frac{h'}{h} e^{\frac{2}{\varepsilon} |s|} dx.$$

Value of $\gamma_2(0)$ by formula (14) is following

$$\gamma_2(0) = \chi_2 \exp \left[2 \int_{x_2}^0 \chi \ln \frac{\chi}{\chi_2} e^{2 \int_0^x \chi dx} dx \right].$$

Introducing in this dominant term of asymptotic of integral (22) we get

$$\gamma_2(0) = \frac{1}{\varepsilon^{\frac{2}{3}}} \sqrt[3]{\frac{3r_1}{4}} e^{-\frac{C}{3}} (1 + O(\varepsilon)). \tag{23}$$

3. Situation corresponding to figure 3, $r_1 < 0$, $x_0 < 0$. We carry out calculations by formula (9) analogical to those corresponding to figure 1.

$$2i \int_{x_0}^0 k \ln \frac{k}{k_0} e^{2i \int_0^x k dx} dx = - \int_{x_0}^0 \left(e^{\frac{2i}{\varepsilon} \int_0^x \sqrt{r} dx} - 1 \right) d \ln \sqrt{r} =$$

 In this case $\sqrt{r} = \sqrt{|r|}$, $s = -|s|$ ($x < 0$) and correlation (19) gives $\sqrt{r} = -\delta^{-1} s^{1/2} h$, so:

$$\begin{aligned}
 &= - \int_{x_0}^0 \left(e^{\frac{2i}{\varepsilon}s(x)} - 1 \right) d \ln \left(-\delta^{-1} s^{1/3} h \right) = \\
 &= - \frac{1}{3} \int_{x_0}^0 \frac{e^{\frac{2i}{\varepsilon}s(x)} - 1}{s(x)} ds(x) - \int_{x_0}^0 \left(e^{\frac{2i}{\varepsilon}s(x)} - 1 \right) \frac{h'(x)}{h(x)} dx = \left. \begin{aligned} t &= -\frac{2}{\varepsilon}s = \frac{2}{\varepsilon}|s| \\ s &= -\frac{\varepsilon}{2}t \\ ds &= -\frac{\varepsilon}{2}dt \end{aligned} \right| = \\
 &= -\frac{1}{3} \int_{\frac{2}{\varepsilon}|s(x_0)|}^0 \frac{e^{-it} - 1}{t} dt + \int_{x_0}^0 \frac{h'}{h} dx - \int_{x_0}^0 e^{\frac{2i}{\varepsilon}s} \frac{h'}{h} dx = \\
 &= \frac{1}{3} \int_0^{\frac{2}{\varepsilon}|s(x_0)|} \frac{e^{-it} - 1}{t} dt - \ln h(x_0) - \int_{x_0}^0 \frac{h'}{h} e^{\frac{2i}{\varepsilon}s} dx = \\
 &= - \left[\ln \left(\sqrt[3]{\frac{2}{\varepsilon} \delta \sqrt{r(x_0)}} \right) + \frac{C}{3} + i \frac{\pi}{6} - O(\varepsilon) \right]. \tag{24}
 \end{aligned}$$

Value $Q(0)$ is calculated by formula (9) as follows

$$Q(0) = k_0 \exp \left[2i \int_{x_0}^0 k \ln \frac{k}{k_0} e^{\frac{2i}{\varepsilon} \int_0^x k dx} dx \right].$$

We substitute in this dominant term of asymptotic of integral (24),

$$Q(0) = \frac{1}{\varepsilon^{2/3}} \sqrt[3]{\frac{3|r_1|}{4}} e^{-\frac{C}{3} - i\frac{\pi}{6}} (1 + O(\varepsilon)). \tag{25}$$

4. Situation corresponding to figure 4, $r_1 < 0, x_1 > 0$. We find value of

$\gamma_1(0)$ from formulae (13). Calculation of integral from formula (13):

$$\begin{aligned}
 & -2 \int_{x_1}^0 \chi \ln \frac{\chi}{\chi_1} e^{-2 \int_0^x \chi dx} dx = - \int_0^{x_1} \ln \frac{\chi}{\chi_1} d \left(e^{-2 \int_0^x \chi dx} - 1 \right) = \\
 & = - \left(e^{-2 \int_0^{x_1} \chi dx} - 1 \right) \ln \frac{\chi}{\chi_1} \Big|_0^{x_1} + \int_0^{x_1} \left(e^{-2 \int_0^x \chi dx} - 1 \right) d \ln \frac{\chi}{\chi_1} = \\
 & = \int_0^{x_1} \left(e^{-\frac{2}{\varepsilon} \int_0^x \sqrt{-r} dx} - 1 \right) d \ln \sqrt{-r} =
 \end{aligned}$$

In this case $\sqrt{-r} = \sqrt{|r|}$, $S = |s|$ and correlation (19) gives $\sqrt{-r} = \delta^{-1} s^{1/3} h$, so:

$$\begin{aligned}
 & = \int_0^{x_1} \left(e^{-\frac{2}{\varepsilon} s} - 1 \right) d \ln \left(\delta^{-1} s^{1/3} h \right) = \\
 & = \frac{1}{3} \int_0^{s(x_1)} \frac{e^{-\frac{2}{\varepsilon} s} - 1}{s} ds + \int_0^{x_1} \left(e^{-\frac{2}{\varepsilon} s} - 1 \right) \frac{h'}{h} dx = \left. \begin{array}{l} t = \frac{2}{\varepsilon} s \\ s = \frac{\varepsilon}{2} t \\ ds = \frac{\varepsilon}{2} dt \end{array} \right| = \\
 & = \frac{1}{3} \int_0^{\frac{2}{\varepsilon} s(x_1)} \frac{e^{-t} - 1}{t} dt - \int_0^{x_1} \frac{h'}{h} dx + \int_0^{x_1} \frac{h'}{h} e^{-\frac{2}{\varepsilon} s} dx = \\
 & = \frac{1}{3} \int_0^{\frac{2}{\varepsilon} s(x_1)} \frac{e^{-t} - 1}{t} dt - \ln h(x_1) + \int_0^{x_1} \frac{h'}{h} e^{-\frac{2}{\varepsilon} s} dx = \\
 & = - \ln \left(\sqrt[3]{\frac{2}{\varepsilon} \delta \sqrt{-r(x_1)}} \right) - \frac{C}{3} + O(\varepsilon). \tag{26}
 \end{aligned}$$

Value of $\gamma_1(0)$ by formula (13) is following

$$\gamma_1(0) = \chi_1 \exp \left[-2 \int_{x_1}^0 \chi \ln \frac{\chi}{\chi_1} e^{-2 \int_0^x \chi dx} dx \right].$$

Substituting in this (26), we get

$$\gamma_1(0) = \frac{1}{\varepsilon^3} \sqrt[3]{\frac{3|r_1|}{4}} e^{-\frac{C}{3}} (1 + O(\varepsilon)). \tag{27}$$

Results of these calculations can be summarized as follows. In turning point $x=0$ ($r(0)=0$) approximate solutions limit values are:

If $r'(0)=r_1 < 0$, then

$$Q(0-0) = R e^{-i\frac{\pi}{6}} (1 + O(\varepsilon)), \tag{28}$$

$$\gamma_1(0+0) = R(1 + O(\varepsilon)), \tag{29}$$

If $r'(0)=r_1 > 0$, then

$$Q(0+0) = R e^{i\frac{\pi}{6}} (1 + O(\varepsilon)), \tag{30}$$

$$\gamma_2(0-0) = R(1 + O(\varepsilon)), \tag{31}$$

where

$$R = \sqrt[3]{\frac{3}{4e^C}} \sqrt[3]{\frac{|r'(0)|}{\varepsilon^2}} = 0,7495 \sqrt[3]{\frac{|r'(0)|}{\varepsilon^2}}. \tag{32}$$

6. CONCLUSION

New asymptotic solution of the Riccati's equation (1) continuous at turning points is represented by the formula (9) (or, in real-valued form – by the formulas (13), (14)). Comparison of expansions (6) and (10) shows that the approximate solution is distinguished from the exact one only in second order of this small parameter. Formula (11) gives power correction in the sense of the argument x . In paragraph 5, we developed the procedures of asymptotic calculation of definite integrals, which have singular dependence from the small parameter in exponential factor. There were obtained principal terms of the asymptotic solutions in turning points. These principal terms (28) – (31) can be used for sewing together various solutions of wave problems with turning points.

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