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PROPERTIES OF PERMUTATION GROUPS USING WREATH PRODUCT

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ABSTRACT

Classification of the p-subgroups of the finite group of order 12 was done using Cauchy's Lagrange's and Sylow's Theorems up to Isomorphism subgroups and related to the Dihedral group of order 20 (D_{2n}) in Chemical Bonding.

Keywords: Isomorphism, Dihedral groups, Chemical bonding, Wreath product, Permutation group, Primitive and transitive.

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1. INTRODUCTION

In Mathematics, the Wreath Product of Group Theory is a specialized product of two groups, based on a semi-direct product. Wreath product is an important tool in the classification of permutation groups and also provides a way of constructing interesting examples of groups. The Wreath product of two groups A and B is written as A w_r B. This is the standard wreath product, for other definitions see [Arbib [1]; Hunter [2]; Kosheler [3] and Nakajima [4]]. The wreath product of A and B contains the direct product A X B as a sub-semi- group. If A has an identity, then any ideal extension of A by B can be imbedded in A w_r B [5]. The question of when A w_r B inherits various properties of A and B has been investigated mainly for various types of simplicity. Some examples are as follows; If A and B are completely-simple semi-groups and A is left-simple, then A w_r B is Completely-Simple [5]. If A and B are semi-groups, with Completely-Simple Kernels, thus A w_r B has a Completely-Simple Kernel [3].

The Wreath product and its generalizations play an important role in the algebraic theory of automata. For example, they can be used to prove the theorem on the decomposition of every finite Semi-group automation into a step wise combination of flip-flope and Simple group automata [2].

2. MATERIALS AND METHODS

2.1 The main objective of this paper is to study under which conditions the wreath products of permutation groups are faithful, transitive and primitive and present an example to support the findings.

2.2 Theorem and Definition: Let A and B be two permutation groups on Γ and Δ respectively. Let A^{Δ} be the set of all maps of Δ into the permutation group, defined by $A^{\Delta} =: \{f: \Delta \to A\}, for\ all\ f_1, f_2 \in A^{\Delta}\ and\ let\ f_1, f_2 \in A^{\Delta}\ be\ defined\ for\ \delta \in \Delta\ by\ (f_1 f_2)\delta =$
$: f_1(\delta_1) f_2(\delta_2) with\ respect\ to\ multiplication, A^{\Delta}\ assume\ the\ structure\ of\ a\ group.$

Proof:

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(i) A^{Δ} is non-empty and is closed with respect to multiplication. For if $f_1, f_2 \in A^{\Delta}, then f_1(\delta_1)f_2(\delta_2) \in A$. Since $f_1(\delta_1)f_2(\delta_2) \in A$, this implies that$(f_1 f_2)\delta \in A \; and \; So \; (f_1 f_2) \in A^{\Delta}$.

(ii) Multiplication in A is associative So also is the multiplication in A^{Δ}.

(iii) The map $\{e: \Delta \rightarrow A\}$ given by $\in(\delta) = 1 \; for \; all \; \delta \in \Delta \; and \; 1 \in C$is the identity element in A^{Δ}.

(iv) For every element $f \in A^{\Delta}$ defined by all $\delta \in \Delta$ by $f^{-1}(\delta) = f(\delta)^{-1}$,thus A^{Δ} is a group under multiplication and we call it G.

Lemma 2.3: Suppose that B acts on G as follows $f^b(\delta) = f(\delta b^{-1}) for \; all \; \delta \epsilon \Delta, \; b \epsilon B, Then \; B \; acts \; on \; G \; as \; agroup.$

Proof: Let $f, f_1, f_2 \in G \; and \; b, b_1, b_2 \in B, Then \; ((f^{b_1})^{b_2})(\delta) = (f^{b_1})\delta b_2^{-1}$

$$= f(\delta b_1^{-1}) \; for \; similarly, f(\delta \, b_2^{-1} b_1^{-1}) = f(\delta(b_1 b_2)^{-1}) = f^{b_1 b_2}(\delta).$$

$$(f_1 f_2 \;)^b(\delta) = f_1 f_2 \; (\delta b^{-1}) = f_1(\delta b^{-1}) f_2 \; (\delta b^{-1}) = f_1{}^b(\delta) f_2{}^b(\delta)$$

$$(f_1, b_1 \;)(f_2, b_2 \;) = (f_1 f_2 \;{}^{b_1^{-1}}, b_1 b_2 \;) \in GB. Hence \; it \; is \; closed.$$

$$Next \; for \; [(f_1, b_1 \;)(f_2, b_2 \;)](f_3, b_3 \;) = (f_1 f_2 \;{}^{b_1^{-1}}, b_1 b_2 \;)(f_3, b_3 \;)$$

$$= (f_1 f_2 \;{}^{b_1^{-1}} f_3^{b_1^{-1} b_2^{-1}}, b_1 b_2 \; b_3) = (f_1 f_2 \;{}^{b_1^{-1}} f_3^{b_2^{-1} b_1^{-1}} \;, b_1 b_2 \; b_3)$$

$$or \; (f_1, b_1 \;)[(f_2, b_2 \;)(f_3, b_3 \;)] = (f_1, b_1 \;)(f_2 f_3 \;{}^{b_2^{-1}}, b_2 b_3 \;)$$

$$= (f_1(f_2 f_3 \;{}^{b_2^{-1}})^{b_1^{-1}}, b_1 b_2 \; b_3) = (f_1 f_2 \;{}^{b_1^{-1}} f_3^{b_2^{-1} b_1^{-1}} \;, b_1 b_2 \; b_3)$$

$$(f, b)(e, 1) = \left(f e^{b^{-1}}, b.1\right) = (f, b) \; or (e, 1)(f, b) = (ef^{-1}, 1.b)$$

$$= (f, b). Hence, B \; acts \; on \; G \; as \; a \; group.$$

From now on, we will not write a group as a map, rather we will use exponential notation. And note that (f,b)=f^b .

For example, the symmetric group on n letters acts naturally on the set {1,2,3,…….,n}. For if n=6, we have $1^{(123)} = 2, 3^{(12)(35)} = 5 \; and \; 6^{(126)(65)(351)} = 3.$

Theorem 2.4: Let B act on G as a group. Thus the set of ordered pairs (f,b) with fε G and bε B thus this Showing that the identity element exists. And thus

$$(f, b)((f^{-1})^b, b^{-1}) = (f(f^{-1})^b)^{b^{-1}}, b^{-1}) = (f(f^{-1})^{bb^{-1}}, bb^{-1}) = (f(f^{-1})^1, 1) = (e, 1)$$

Hence, the inverse also exists and thus B acts on the G as a group if the set of all ordered pairs (f,b) if we define

$(f_1, b_1)(f_2, b_2) = \left(f_1 f_2{}^{b_1{}^{-1}}, b_1 b_2\right)$.

Definition 2.5: Let A and B be groups then the wreath of A by B denoted by $W = A \; wr \; B$ is the semi-direct product of G by B, so that $W = \{(f, b) \setminus f \epsilon G, b \epsilon B\}$, with multiplication in W defined as $(f_1, b_1)(f_2, b_2) = \left((f_1 f_2{}^{b_1{}^{-1}}), (b_1 b_2)\right)$ for all $f_1, f_2 \epsilon G \; and \; b_1 b_2 \; \epsilon \; B$.

2.6 Orbits and Stabilisers: The two most important concepts in the theory of group actions are that of an orbit and the stabiliser. An orbit can be thought of as what you get when you spin an element under the action of G [John Bamberg]

Definition 2.7: Let B be a group acting on a set Δ , and let δ under B is the subset of Δ defined $\delta^B := \{\delta^b : b \; \epsilon \; B\}$.

For example, the orbit of 1 under the subgroup $(1 \;\; 2 \;\; 3), (2 \;\; 3)(4 \;\; 5)$ of S_5 , is $\{1, \; 2, \; 3\}$.

Definition 2.8: The stabiliser of any point $(\alpha, \delta) \; in \, \Gamma \times \Delta$ under the action of W on $\Gamma \times \Delta$ denoted by $W_{(\alpha,\delta)}$ is given by $W_{(\alpha,\delta)} = \{fb \epsilon W \setminus (\alpha, \delta) fb = (\alpha, \delta)\} = \{fb \epsilon W \setminus (\alpha f(\delta), \delta b) = (\alpha, \delta)\} = \{fb \epsilon W \setminus \alpha \; f(\delta) = \alpha, \delta b = \delta\}$

$$= F(\delta)_\alpha B_\delta$$

Definition 2.9: For any two points (α_1, δ_1) and (α_2, δ_2) in $\Gamma \times \Delta$, then W will be transitive on $\Gamma \times \Delta$ if and only if $fb\epsilon W, f\epsilon G, b\epsilon B$ such that $(\alpha_1, \delta_1)fb = (\alpha_1 f(\delta_1), \delta_1 b) = (\alpha_2, \delta_2)$. Thus such f, b exist if A and B are transitive on Γ *and* Δ respectively which is a necessary condition for W to be transitive on $\Gamma \times \Delta$.

Definition 2.10: W is faithful on $\Gamma \times \Delta$ if and only if the identity element of W is its only element that fixes every point of $\Gamma \times \Delta$ or W is faithful on $\Gamma \times \Delta$, if A and B are faithful on Γ *and* Δ respectively.

Definition2.11: W is primitive on $\Gamma \times \Delta$, if and only if given any (α, δ) in $\Gamma \times \Delta$, $W_{(\alpha,\delta)}$ the stabiliser of (α, δ) is a maximal subgroup of W. That is, if $W_{(\alpha,\delta)} \neq W \text{ and } W_{(\alpha,\delta)} \underline{\Delta}\, U_{(\alpha,\delta)} \underline{\Delta}\, W$ implies either $W_{(\alpha,\delta)} = U_{(\alpha,\delta)} or\ U_{(\alpha,\delta)} = W$.

Definition2.12: Let G be a group acting on Ω, A block for G is a subset $N \subset \Omega$ *with* $|N| > 1$ *and* $N \neq \Omega$ but that for all $g\epsilon G, N^g \cap N = \emptyset \text{ or } N^g = N$. In other words W is imprimitive if transitive and there are blocks. Generally for $(Z_n, +)$ *on* $\{0,1,2,\dots,n-1\}$, *if* $k \setminus n$ *and* $r\epsilon Z$ *then* $\{i\epsilon\Omega: i = r \bmod k\}$ is a block. Example, taking n=6, then $\{i\epsilon\Omega: i = 2 \bmod 3\} = \{2, 5\}$ is a block.

Theorem 2.13: The centre of W written Z(w) is defined where $k = \{b\epsilon B \setminus \delta b = b \text{ for all } \delta\epsilon\Delta\}$ $by\ Z(w) = fb \setminus (fb)(f_1, b_1) = (f_1, b_1)(fb);\ for\ all\ f_1\, \epsilon G, b_1 \epsilon B\}$. For $fb\epsilon\, Z(w)$if and only If

$$ff_1^{b^{-1}}bb_1 = f_1 f_2^{b_1^{-1}} b_1 b \text{ for all } f_1\epsilon G, b\epsilon B \dots\dots\dots\dots\dots\dots\dots\dots\dots (1.1)$$

Proof:

Substituting $b_1 = 1$, (1.1) becomes $ff_1^{b^{-1}}b = f_1 f_2 b \text{ for all } f_1 \epsilon G \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.2)$

Putting $f_1 = 1$ in (1.1) we get $fbb_1 = f_2^{b_1^{-1}} b_1 b \text{ for all } b_1 \epsilon B \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.3)$

Therefore, for $fb\epsilon\, Z(w)$, then $b\epsilon\, Z(B)$.

<u>Claim</u>:

If $A \neq 1, fb\epsilon\, Z(w) \text{ and } b\epsilon\, Z(B), \text{then } \delta b = \delta \text{ for all } \delta\, \epsilon\, \Delta \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.4)$

Choose $f_1 \epsilon\ G\ such\ that$

$$f_1(\delta) = c \neq 1, a \in A \text{ and } f_1(\delta') = 1, \forall\, \delta' \neq \delta \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.5)$$

And so, from (1.2) we have $f_1(\delta)f(\delta) = f(\delta)f_1(\delta b) = f(\delta), if\ \delta b \neq \delta$ showing that $f_1(\delta) = 1$ is a contradiction from equation (1.5). Thus $\delta b = \delta$ for all $\delta \in \Delta$. Hence our claim. But (1.2) also implies that for all $\delta\epsilon\Delta, f_1(\delta)f(\delta) = f(\delta)f_1(\delta b) = f(\delta)f_1(\delta)$. So $f(\delta) \in Z(A)$, for all $\delta \in \Delta \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.6)$

But (1.3) implies that $f_1(\delta b_1) = f(\delta) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.7)$

Since f is constant over orbits of B in Δ and from (1.4) , (1.6) and (1.7) we conclude that for $A \neq \{1\}, fb \in Z(w)$ if and only if

$$Z(w) = \begin{cases} Z(B) & if\ A = 1 \\ (\pi Z_i)(Z(B) \cap k), & otherwise \end{cases}$$

3. RESULTS AND DISCUSSION

Example: Let the permutation group $\frac{C}{A}= \{(1), (1\ 2\ 4), (1\ 4\ 2)\}$ and $\frac{D}{B}= \{(1), (5\ 6)\}$ act on the sets $\Gamma = \{1,2,4\}$ and$\Delta = \{5,6\}$ respectively. Also let $G = A^{\Delta} = \{f \setminus \Delta \to A\}$, then $|P| = |A|^{|\Delta|} = 3^2 = 9$. The mappings are as follows:

$$f_1: 5 \to (1), \quad 6 \to (1)$$
$$f_2: 5 \to (1\ 2\ 4), \quad 6 \to (1\ 2\ 4)$$
$$f_3: 5 \to (1\ 4\ 2), \quad 6 \to (1\ 4\ 2)$$
$$f_4: 5 \to (1), \quad 6 \to (1\ 2\ 4)$$
$$f_5: 5 \to (1), \quad 6 \to (1\ 4\ 2)$$
$$f_6: 5 \to (1\ 2\ 4), \quad 6 \to (1)$$
$$f_7: 5 \to (1\ 4\ 2), \quad 6 \to (1)$$
$$f_8: 5 \to (1\ 2\ 4), \quad 6 \to (1\ 4\ 2)$$

$$f_9: 5 \to (1\ 4\ 2), \qquad 6 \to (1\ 2\ 4)$$

With respect to the operators $(f_1 f_2)(\delta) = f_1(\delta) f_2(\delta)$ it can be verified that G is a group for $\delta \epsilon \Delta$. We define the action of B on G as $f^b(\delta) = f(\delta b^{-1}), \forall\ g \in b \in B, \delta \epsilon \Delta$, then B acts on G as groups.

Define $W = A\ wr\ B$, the semi-direct product of G by B in that order; that $W = \{fb: f \in G, b \in B\}$ so that W is a group with respect to the operation $(f_1 b_1)(f_2 b_2) = (f_1 f_2^{{b_1}^{-1}})(\ b_1 b_2)$ so that our $b_1 = (1)$ and $b_2 = (4\ 5)$. Thus this will give the element of W as $(f_1 b_1), (f_2 b_1), (f_3 b_1), (f_4 b_1), (f_5 b_1), (f_6 b_1), (f_7 b_1), (f_8 b_1), (f_9 b_1), (f_1 b_2), (f_2 b_2), (f_3 b_2),$
$(f_4 b_2). (f_5 b_2), (f_6 b_2), (f_7 b_2), (f_8 b_2), (f_9 b_2)$.

Next we define our W on $\Gamma \times \Delta$ as shown below;$(\alpha, \delta) fb = (\propto f(\delta), \delta b)$. But also $\Gamma \times\ \Delta = \{(1,5), (1,6), (2,5), (2,6), (4,5), (4,6)\}$.

And the following permutations by the action of W on $\Gamma \times\ \Delta$

$$(\Gamma \times\ \Delta) f_1 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_2 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (2,5)(2,6)(4,5)(4,6)(1,5)(1,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_3 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (4,5)(4,6)(1,5)(1,6)(2,5)(2,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_4 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (1,5)(2,6)(2,5)(4,6)(4,5)(1,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_5 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (1,5)(4,6)(2,5)(1,6)(4,5)(4,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_6 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (2,5)(1,6)(4,5)(2,6)(1,5)(4,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_7 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (4,5)(1,6)(1,5)(2,6)(2,5)(4,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_8 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (2,5)(4,6)(4,5)(1,6)(1,5)(2,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_9 b_1 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (4,5)(2,6)(1,5)(4,6)(2,5)(1,6) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_1 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (1,6)(1,5)(2,6)(2,5)(4,6)(4,5) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_2 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (2,6)(2,5)(4,6)(4,5)(1,6)(1,5) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_3 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (4,6)(4,5)(1,6)(1,5)(2,6)(2,4) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_4 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (1,6)(2,5)(2,6)(4,5)(4,6)(1,5) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_5 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (1,6)(4,5)(2,6)(1,5)(4,6)(2,5) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_6 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (2,6)(1,5)(4,6)(2,5)(1,6)(4,5) \end{pmatrix}$$

$$(\Gamma \times\ \Delta) f_7 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (4,6)(1,5)(1,6)(2,5)(2,6)(4,5) \end{pmatrix}$$

$$(\Gamma \times \ \Delta)f_8 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (2,6)(4,5)(4,6)(1,5)(1,6)(2,5) \end{pmatrix}$$

$$(\Gamma \times \ \Delta)f_9 b_2 = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (4,6)(2,5)(1,6)(4,5)(4,6)(1,5) \end{pmatrix}$$

If we rename the symbols for convenience as $(1,5) = 1$, $(1,6) = 2$, $(2,5) = 3$, $(2,6) = 4$, $(4,5) = 5$, $(4,6) = 6$. And so in cyclic form, we write these permutations as: (1), $(1\ 3\ 5)(2\ 4\ 6)$, $(1\ 5\ 3\ )(2\ 6\ 4)$, $(2\ 4\ 6)$, $(2\ 6\ 4)$, $(1\ 3\ 5)$, $(1\ 5\ 3)$, $(1\ 3\ 5)(2\ 6\ 4)$, $(1\ 5\ 3\ )(2\ 4\ 6)$, $(1\ 2)(3\ 4\ )(5\ 6\ )$, $(1\ 4\ 5\ 2\ 3\ 6)$, $(1\ 6\ 3\ 2\ 5\ 4)$, $(1\ 2\ 3\ 4\ 5\ 6)$, $(1\ 6\ 5\ 4\ 3\ 2)$, $(1\ 2\ 5\ 6\ 3\ 4\ )$, $(1\ 4\ 3\ 6\ 5\ 2\ )$, $(1\ 4\ )(2\ 5\ )(3\ 6\ )$

$$and \quad (1\ 6)(2\ 3)(4\ 5)$$

3. CONCLUSION

The research paper studied and presented the conditions in which the wreath products of permutation groups prove to be faithful, transitive and primitive. An example was presented and used as an illustration to support the findings.

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REFERENCES

[1] M. A. Arbib, *Algebraic theory of machines, languages and Semi-groups*. New York: Academic Press, 1968.

[2] R. P. Hunter, "Some results on wreath products of Semi-groups," *Bull Soc. Math. Belgique*, vol. 18, pp. 3-16, 1966.

[3] Y. G. Kosheler, "Wreath products and equations in Semi-groups," *Semi-Groups Forum*, vol. 11, pp. 1-13, 1975.

[4] S. Nakajima, "On the Kernel of the wreath product of completely simple Semi-groups 11," presented at the First Symp., 1977.

[5] K. Krohn and J. Rhodes, "Algebraic theory of machines. Prime decomposition theorem for finite semi- groups and machines," *Trans. American Math. Soc.*, vol. 116, pp. 450-464, 1965.

BIBLIOGRAPHY

[1] J. J. D. McKnight and E. Sadowski, "The Kernel of the wreath product of Semi-groups," *Semi-Group Forum*, vol. 4, pp. 232-236, 1972.

[2] B. Meenaxi, R. G. Moller, D. Macpherson, and P. M. Neumann, *Notes on infinite permutation groups*. New Delhi: Hindustan Book Agency, 1997.

[3] B. H. Nenmann, "Embedding theorems for Semi-groups," *J. London Mathematical Society*, vol. 35, pp. 184-192, 1960.