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THE (F/G)-EXPANSION METHOD AND TRAVELLING WAVE SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS

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ABSTRACT

The (F/G)-expansion method is firstly proposed, where F=F(ξ) and $G = G(\xi)$ satisfies a first order ordinary differential equation systems (ODEs). We give the exact travelling wave solutions of the variant Boussinesq equations and the KdV equation and by using (F/G)-expansion method. When some parameters of present method are taken as special values, results of the (G'/G)-expansion method are also derived. Hence, (G'/G)-expansion method is sub method of the proposed method. The travelling wave solutions are expressed by three types of functions, which are called the trigonometric functions, the rational functions, and the hyperbolic functions. The present method is direct, short, elementary and effective, and is used for many other nonlinear evolution equations.

Keywords: Nonlinear, (F/G)- expansion, Homogeneous balance, Travelling wave, KdV equation, Variant boussinesq.

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Contribution/ Originality

This study contributes in the existing literature of (G'/G)-expansion method. We proposed the (F/G)-expansion method and investigated the exact travelling wave solutions of the variant Boussinesq equations and the KdV equation by using (F/G)-expansion method.

1. INTRODUCTION

The nonlinear phenomena are frequently encountered in all the fields, including either the scientific work or engineering fields, such as fluid mechanics, optical fibers, plasma physics, solid state physics, biology, chemical kinematics, chemical physics. To define these complex phenomena, many non-linear evolution equations (NLEEs) are widely used. During the past four decades or so, the many researchers are interested to find powerful and efficient methods for analytic solutions of nonlinear equations. Many powerful methods to obtain exact solutions of nonlinear evolution equations have been constricted and developed such as the inverse scattering transform in [1], the Backlund/Darboux transform in [2-4], the Hirota's bilinear operators in [5], the truncated Painleve expansion in [6], the tanh-function expansion and its various extension in [7-9], the Jacobi elliptic function expansion in [10, 11], the F-expansion in [12-15], the sub-ODE method in [16-19], the homogeneous balance method in [20-22], the sine–cosine method in [23, 24] the rank analysis method in [25], the ansatz method in [26-28], the exp-function expansion method in [29] and so on, but there is no unified method that can be used to deal with all types of nonlinear evolution equations.

Recently, Wang, et al. [30] introduced a new direct method called (G'/G)-expansion method for the NLEEs. The value of the (G'/G)-expansion method is that one can deal with nonlinear problems by essentially linear methods. In the method, nonlinear evolution equations are transformed to the explicit linear differential equations for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. The (G'/G)-expansion method is widely used by many authors [31, 32]. More recently, this method has been generalized by Zhang, et al. [33]; Zhang, et al. [34] to obtain non-traveling wave solutions and coefficient function solutions. Later, Zhang, et al. [35] further extend the method to obtain solution of an evolution equation with variable coefficients. Also, Zhong, et al. [36] designed an algorithm for using the method to obtain solution of nonlinear differential-difference equations. Then, Yu-Bin and Chao [37] adapted the method to get traveling wave solutions for Witham-Broer-kaup-Like equations.

In the present paper, we shall propose a new method, which is called the (F/G)-expansion method to obtain travelling wave solutions of nonlinear evolution equations. In the method, nonlinear evolution equations are transformed to the explicit linear differential equations for traveling waves with a certain substitution which leads to a system of first-order differential equation with constant coefficients. The main idea of the given method is that the travelling wave solution of a nonlinear evolution equation is expressed by a polynomial in (F/G). Where $F=F(\xi)$ and $G=G(\xi)$ are solutions of a system of first order ODEs and $\xi=x-ct$. The degree of the polynomial of the (F/G) can be decided by considering the homogeneous balance between the highest order derivatives and nonlinear terms which is appearing in nonlinear evolution equation. The coefficients of the polynomial can be obtained if system of algebraic equations which is obtained by using the proposed method can be solved. It will be shown that more travelling wave solutions of many nonlinear evolution equations can be acquired by using the (F/G)-expansion method.

In (F/G)-expansion method, it is considered that $F=F(\xi)$ and $G=G(\xi)$ are satisfied system of first order ODEs in the form

$$\begin{cases} F'+aF+eG+bFG=0 \\ G'+cF+dG+bG^2=0 \end{cases}$$

for $e=0$. If we consider $b=0$, $c=-1$, $d=0$ and $e\neq 0$ in the above a system of first order ODEs, this method is transformed to the (G'/G)-expansion method. Hence, (G'/G)-expansion method is special form of the present method.

In Section 2, we give an account of the (F/G)-expansion method for obtaining travelling wave solutions of nonlinear evolution equations. The principal steps of the present method are given in here. In the subsequent sections, the proposed method is applied to the celebrated KdV equation and the variant Boussinesq equations. In the last section, the features of the (F/G)-expansion method are briefly summarized.

2. DESCRIPTION OF THE (F/G)-EXPANSION METHOD

In this part, we describe the (F/G)-expansion method for obtaining travelling wave solutions of nonlinear evolution equations. Suppose that

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0 \tag{2.1}$$

is a nonlinear equation of $u(x,t)$. Where $u = u(x,t)$ is an unknown function, *P* is a polynomial of $u = u(x,t)$ and it's various partial derivatives with respect to *x* and *t*, in which nonlinear terms and the highest order derivatives are involved. Principal steps of the (F/G)-expansion method will be given as follows:

Step 1. Combining the independent variables *x* and *t* into one variable $\xi = x - Vt$, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x - Vt, \tag{2.2}$$

The travelling wave variable in (2.2) allows us reducing (2.1) to an ODE for $u = u(\xi)$

$$P(u, -Vu', u', V^2u'', -Vu'', u'', \dots) = 0 \tag{2.3}$$

Step 2. We consider that the solution of equation (2.3) can be taken by a polynomial in (F/G) as follows:

$$u(\xi) = \sum_{m=0}^{n} \alpha_m (F/G)^m \tag{2.4}$$

Where $F = F(\xi)$ and $G = G(\xi)$ satisfy the following system of equation

$$\begin{cases} F' + aF + bFG = 0 \\ G' + cF + dG + bG^2 = 0 \end{cases} \tag{2.5}$$

$\alpha_0, \alpha_1, \dots, \alpha_n, a, b, c, d$ and V are constants to be determined later, $\alpha_n \neq 0$ and, n is a positive integer, which can be determined by considering the homogeneous balance between nonlinear terms and the highest order derivatives appearing in Eq. (2.3).

Step 3. By replacing (2.4) into (2.3), using equation (2.5) and gathering all terms of the same order of (F/G) together, the left-hand side of Eq. (2.3) is organized as polynomial of (F/G). If each coefficients of obtained polynomial are equating to the zero, algebraic equations system for $\alpha_0, \alpha_1, \dots, \alpha_n, a, b, c, d$ and V can be obtained

Step 4. The constants $\alpha_0, \alpha_1, \dots, \alpha_n$ and V can be obtained by solving the algebraic equations in Step 3. The general solutions of the system of first order ODEs (2.5) have been well known for us depending on the sign of the discriminant $\Delta = d^2 + 4ac$. We have more travelling wave solutions of the nonlinear evolution equation (2.1) by substituting $\alpha_1, \alpha_2, \dots, \alpha_n, V$ and the general solutions of (2.5) into (2.4).

In the subsequent sections we will give examples such as celebrated KdV equation and the variant Boussinesq equations for proposed method.

3. KDV EQUATION

In this part, we apply our proposed method to three celebrated KdV equation in the form

$$u_t + uu_x + \delta u_{xxx} = 0, \tag{3.1}$$

which arises in many physical problems such as ion-acoustic waves in plasma and surface water waves. The variable of travelling wave given as:

$$u(x,t) = u(\xi), \quad \xi = x - Vt \tag{3.2}$$

Allows us transforming (3.1) into an ODE for $u = u(\xi)$

$$-Vu' + uu' + \delta u''' = 0.$$

If this equation is integrated with respect to ξ, equation given as:

$$C - Vu + \frac{1}{2}u^2 + \delta u'' = 0, \tag{3.3}$$

can be acquired. Where C is integration constant which will be determined in the below.

Suppose that the solution of equation (3.3) can be given by a polynomial of (F/G) as follows:

$$u(\xi) = \sum_{m=0}^{n} \alpha_m (F/G)^m \tag{3.4}$$

Where $F = F(\xi)$ and $G = G(\xi)$ satisfy a system of first order ODEs in the form

$$\begin{cases} F' + aF + bFG = 0 \\ G' + cF + dG + bG^2 = 0 \end{cases} \tag{3.5}$$

If we consider homogeneous balance between u'' and u^2 in (3.3), we can obtain $2n = n + 2 \Rightarrow n = 2$. Hence we can write (3.4) as

$$u(\xi) = \alpha_2 (F/G)^2 + \alpha_1 (F/G) + \alpha_0 \tag{3.6}$$

And also

$$u^2(\xi) = \alpha_2^2 (F/G)^4 + 2\alpha_2\alpha_1 (F/G)^3 + (\alpha_1^2 + 2\alpha_2\alpha_0)(F/G)^2 + 2\alpha_1\alpha_0 (F/G) + \alpha_0^2 \quad (3.7)$$

By using (3.6) and (3.5), it is easily obtained that

$$u''(\xi) = 6c^2\alpha_2 (F/G)^4 + (10cd\alpha_2 + 2c^2\alpha_1)(F/G)^3 + ((4d^2 - 8ac)\alpha_2 + 3cd\alpha_1)(F/G)^2 \quad (3.8)$$
$$+ (-6ad\alpha_2 + (d^2 - 2ac)\alpha_1)(F/G) + 2a^2\alpha_2 - ad\alpha_1$$

If we substitute (3.6), (3.7) and (3.8) into (3.3) and all terms of same power of (F/G) are collected together, the left-hand side of (3.3) is turned into polynomial in (F/G). If each coefficients of this polynomial are equated to zero, we can obtain a algebraic equations system of $\alpha_0, \alpha_1, \alpha_2, a, b, c, d, V$ and C as follows:

$$(F/G)^0: \quad 2a^2\delta\alpha_2 - ad\delta\alpha_1 + \frac{\alpha_0^2}{2} - V\alpha_0 + C = 0$$

$$(F/G)^1: \quad -6ad\delta\alpha_2 + (d^2\delta - 2ac\delta - V)\alpha_1 + \alpha_1\alpha_0 = 0$$

$$(F/G)^2: \quad (4d^2\delta - 8ac\delta - V)\alpha_2 + 3cd\delta\alpha_1 + \frac{\alpha_1^2}{2} + \alpha_2\alpha_0 = 0$$

$$(F/G)^3: \quad 2c^2\delta\alpha_1 + 10cd\delta\alpha_2 + \alpha_2\alpha_1 = 0$$

$$(F/G)^4: \quad \frac{\alpha_2^2}{2} + 6c^2\delta\alpha_2 = 0$$

Solving the above system, we can get the following set of solutions.

$$\alpha_2 = -12c^2\delta, \quad \alpha_1 = -12cd\delta, \quad V = d^2\delta - 8ac\delta + \alpha_0 \quad (3.9)$$
$$C = \frac{\alpha_0^2}{2} + (d^2 - 8ac)\delta\alpha_0 + 24a^2c^2\delta^2 - 12acd^2\delta^2$$

Where α_0, a, c, d and δ are arbitrary constants.

If we substitute the general solutions of ordinary differential equation system of (3.5), $\xi = x - (d^2\delta - 8ac\delta + \alpha_0)t$ and coefficients (3.9) into (3.6), we can get three types of travelling wave solutions of the KdV equation (3.1) as follows:

When $4ac + d^2 > 0$, F/G is acquired as

$$F/G=\frac{2a\left(K_1\cosh\frac{\sqrt{d^2+4ac}}{2}\xi+K_2\sinh\frac{\sqrt{d^2+4ac}}{2}\xi\right)}{\left(K_1d-K_2\sqrt{d^2+4ac}\right)\cosh\frac{\sqrt{d^2+4ac}}{2}\xi+\left(K_2d-K_1\sqrt{d^2+4ac}\right)\sinh\frac{\sqrt{d^2+4ac}}{2}\xi}$$

If it is regulated, it can be shown as

$$F/G=\frac{2a}{d-\tanh\left(\frac{\sqrt{4ac+d^2}(\xi-C_1)}{2}\right)\sqrt{4ac+d^2}}$$

Hence, solution of the KdV equation (3.1) is acquired as follows:

$$u_1(\xi)=-12c^2\delta\left(\frac{2a}{d-\tanh\left(\frac{\sqrt{4ac+d^2}(\xi-C_1)}{2}\right)\sqrt{4ac+d^2}}\right)^2$$
$$-12cd\delta\left(\frac{2a}{d-\tanh\left(\frac{\sqrt{4ac+d^2}(\xi-C_1)}{2}\right)\sqrt{4ac+d^2}}\right)+\alpha_0$$

Where $\xi=x-(d^2\delta-8ac\delta+\alpha_0)t$ and C_1 (or K_1 and K_2) is arbitrary constant.

When $4ac+d^2=0$, F/G is acquired as

$$F/G=\frac{2a(K_2\xi-K_1)}{d(K_2\xi+K_1)-2K_2}=\frac{2a(\xi+C_1)}{d(\xi+C_1)-2}$$

Hence, solution of the KdV equation (3.1) is acquired as follows:

$$u_2(\xi)=-12c^2\delta\left(\frac{2a(\xi+C_1)}{d(\xi+C_1)-2}\right)^2-12cd\delta\left(\frac{2a(\xi+C_1)}{d(\xi+C_1)-2}\right)+\alpha_0$$

where $\xi=x-(d^2\delta-8ac\delta+\alpha_0)t$ and C_1 (or K_1 and K_2) is arbitrary constant.

When $4ac+d^2<0$, F/G is obtained as

$$F/G=\frac{2a\left(K_1\cos\frac{\sqrt{-d^2-4ac}}{2}\xi+K_2\sin\frac{\sqrt{-d^2-4ac}}{2}\xi\right)}{\left(K_1d-K_2\sqrt{-d^2-4ac}\right)\cos\frac{\sqrt{-d^2-4ac}}{2}\xi+\left(K_2d+K_1\sqrt{-d^2-4ac}\right)\sin\frac{\sqrt{-d^2-4ac}}{2}\xi}$$

If it is regulated, it can be shown as

$$F/G = \frac{2a}{d + \tan\left(\frac{\sqrt{-(4ac+d^2)}(\xi - C_1)}{2}\right)\sqrt{-(4ac+d^2)}}$$

Hence, solution of the KdV equation (3.1) is acquired as follows:

$$u_3(\xi) = -12c^2\delta\left(\frac{2a}{d + \tan\left(\frac{\sqrt{-(4ac+d^2)}(\xi - C_1)}{2}\right)\sqrt{-(4ac+d^2)}}\right)^2 -12cd\delta\left(\frac{2a}{d + \tan\left(\frac{\sqrt{-(4ac+d^2)}(\xi - C_1)}{2}\right)\sqrt{-(4ac+d^2)}}\right) + \alpha_0$$

Where $\xi = x - (d^2\delta - 8ac\delta + \alpha_0)t$ and C_1 (or K_1 and K_2) is arbitrary constant.

4. VARIANT BOUSSINESQ EQUATIONS

In this part, our proposed method have been applied to the variant Boussinesq equations given in the form

$$\begin{cases} H_t + (Hu)_x + u_{xxx} = 0 \\ u_t + H_x + uu_x = 0 \end{cases} \quad (4.1)$$

This is a model for water waves. Where $u = u(x,t)$ represents the velocity and $H = H(x,t)$ represents total depth. The variable of travelling wave given as:

$$H(x,t) = H(\xi),\ u(x,t) = u(\xi),\quad \xi = x - Vt \quad (4.2)$$

allows us transforming (4.1) into ODEs for $u = u(\xi)$ and $H = H(\xi)$ as follows:

$$\begin{cases} -VH' + (Hu)' + u''' = 0 \\ -Vu' + H' + uu' = 0 \end{cases}$$

If we integrate the ODEs above with respect to ξ, we can get equation given as:

$$\begin{cases} C_1 - VH + Hu + u'' = 0 \\ C_2 - Vu + H + \frac{u^2}{2} = 0 \end{cases} \quad (4.3)$$

Where C_1 and C_2 are integration constants whose values will be determined in the below.

If we consider homogeneous balance between u'' and Hu in first equation of (4.3) and also between H and u^2 in second equation of (4.3), we obtain

$$n_1 + n_2 = n_2 + 2,\ \ 2n_2 = n_1 \Rightarrow n_1 = 2,\ n_2 = 1$$

Similar to (3.4), we conjecture that

$$H(\xi) = \alpha_2 (F/G)^2 + \alpha_1 (F/G) + \alpha_0 \tag{4.4}$$

$$u(\xi) = +\beta_1 (F/G) + \beta_0 \tag{4.5}$$

Where $F = F(\xi)$ and $G = G(\xi)$ satisfy (3.5) and $\alpha_2,\ \alpha_1,\ \alpha_0,\ \beta_1$ and β_0 are constants whose values will be determined in the below.

By using (4.4), (4.5) and (3.5), it is easily to get that

$$Hu = \alpha_2\beta_1 (F/G)^3 + (\alpha_1\beta_1 + \alpha_2\beta_0)(F/G)^2 + (\alpha_0\beta_1 + \alpha_1\beta_0)(F/G) + \alpha_0\beta_0 \tag{4.6}$$

$$u'' = 2\beta_1 c^2 (F/G)^3 + 3\beta_1 cd (F/G)^2 + (\beta_1 d^2 - \beta_1 ac)(F/G) - \beta_1 ad \tag{4.7}$$

$$u^2 = \beta_1^2 (F/G)^2 + 2\beta_0\beta_1 (F/G) + \beta_0^2 \tag{4.8}$$

If we substitute (4.6), (4.7) and (4.8) into (4.3) and all terms of same power of (F/G) are collected together, the left-hand side of (3.3) is turned into polynomial in (F/G). If each coefficients of this polynomial are equated to zero, we can obtain an algebraic equations system of $\alpha_0,\ \alpha_1, \alpha_2,\ \beta_1,\ \beta_0,\ a,\ b,\ c, d, V,\ C_1$ and C_2 as follows:

$$(F/G)^0: \qquad \alpha_0\beta_0 - \beta_1 ad - V\alpha_0 + C_1 = 0$$

$$(F/G)^1: \qquad -V\alpha_1 + \alpha_0\beta_1 + \alpha_1\beta_0 + -2\beta_1 ac + \beta_1 d^2 = 0$$

$$(F/G)^2: \qquad -V\alpha_2 + 3cd\beta_1 + \alpha_1\beta_1 + \alpha_2\beta_0 = 0$$

$$(F/G)^3: \qquad 2c^2\beta_1 + \alpha_2\beta_1 = 0$$

$$(F/G)^0: \qquad C_2 - V\beta_0 + \frac{\beta_0^2}{2} + \alpha_0 = 0$$

$$(F/G)^1: \qquad \alpha_1 + \beta_0\beta_1 - V\beta_1 = 0$$

$$(F/G)^2: \qquad \alpha_2 + \frac{\beta_1^2}{2} = 0$$

Solving the above system, the following set of solutions can be acquired.

$$\alpha_2 = -2c^2, \quad \alpha_1 = -2cd, \quad \alpha_0 = 2ac,$$

$$\beta_1 = \mp 2c, \quad V = \beta_0 \pm d \qquad (4.9)$$

$$C_1 = 0, \quad C_2 = \frac{\beta_0^2}{2} \pm \beta_0 d - 2ac$$

Where β_0, a, c and d are arbitrary constants.

If we substitute the general solutions of ordinary differential equation system of (3.5), coefficients in equation (49) and $\xi = x - (\beta_0 \pm d)t$ into (4.4) and (4.5), we can get three types of travelling wave solutions of the variant Boussinesq equations (4.1) as follows:

When $4ac + d^2 > 0$, solutions of variant Boussinesq equations are acquired as

$$H_1(\xi) = -2c^2\left(\frac{2a}{d - \tanh\left(\frac{\sqrt{4ac+d^2}(\xi - K_1)}{2}\right)\sqrt{4ac+d^2}}\right)^2 - 2cd\left(\frac{2a}{d - \tanh\left(\frac{\sqrt{4ac+d^2}(\xi - K_1)}{2}\right)\sqrt{4ac+d^2}}\right) + 2ac$$

$$u_1(\xi) = \mp 2c\left(\frac{2a}{d - \tanh\left(\frac{\sqrt{4ac+d^2}(\xi - K_1)}{2}\right)\sqrt{4ac+d^2}}\right) + \beta_0$$

Where $\xi = x - (\beta_0 \pm d)t$ and K_1 is arbitrary constant.

When $4ac + d^2 = 0$, solutions of variant Boussinesq equations are acquired as

$$H_2(\xi) = -2c^2\left(\frac{2a(\xi + K_1)}{d(\xi + K_1) - 2}\right)^2 - 2cd\left(\frac{2a(\xi + K_1)}{d(\xi + K_1) - 2}\right) + 2ac$$

$$u_2(\xi) = \mp 2c\left(\frac{2a(\xi + K_1)}{d(\xi + K_1) - 2}\right) + \beta_0$$

Where $\xi = x - (\beta_0 \pm d)t$ and K_1 is arbitrary constant.

When $4ac+d^2<0$, solutions of variant Boussinesq equations are acquired as

$$H_3(\xi)=-2c^2\left(\frac{2a}{d+\tan\left(\frac{\sqrt{-(4ac+d^2)}(\xi-K_1)}{2}\right)\sqrt{-(4ac+d^2)}}\right)^2$$

$$-2cd\left(\frac{2a}{d+\tan\left(\frac{\sqrt{-(4ac+d^2)}(\xi-K_1)}{2}\right)\sqrt{-(4ac+d^2)}}\right)+2ac$$

$$u_3(\xi)=\mp 2cd\left(\frac{2a}{d+\tan\left(\frac{\sqrt{-(4ac+d^2)}(\xi-K_1)}{2}\right)\sqrt{-(4ac+d^2)}}\right)+\beta_0$$

Where $\xi=x-(\beta_0\pm d)t$ and K_1 is arbitrary constant.

5. CONCLUSIONS

In this study, the (F/G)-expansion method is firstly proposed. We have shown that three types of travelling solutions of the variant Boussinesq equations and the KdV equation are successfully found out by using the (F/G)-expansion method. When some of the parameters of present method are taken as special values, results of the (G'/G)-expansion method is also obtained automatically. Hence (G'/G)-expansion method is sub method of the (F/G)-expansion method. Applications of this method are very easy, direct, concise, elementary and effective. The present method can be applied further works to establish completely new solutions for other kinds of nonlinear wave equations for instance the Burgers equation [38] the KdV– Burgers equation [21] the Boussinesq equation [1] the Gardner equation [39] the Sharma–Tasso–Olver equation [40] the generalized KPP equation [41] the approximate long water wave equations [22] and the Broer–Kaup equations [42] and so on.

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REFERENCES

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering transform*. Cambridge: Cambridge Univ. Press, 1991.

[2] C. Rogers and W. F. Shadwick, *Backlund transformations*. New York: Academic Press, 1982.

[3] M. Wadati, H. Sanuki, and K. Konno, "Relationships among inverse method, bäcklund transformation and an infinite number of conservation laws," *Prog. Theor. Phys.*, vol. 53, pp. 419-436, 1975.

[4] V. A. Matveev and M. A. Salle, *Darboux transformation and solitons*. Berlin: Springer, 1991.

[5] R. Hirota, "Exact envelope soliton solutions of a nonlinear wave equation," *J. Math. Phys.*, vol. 14, pp. 805-810, 1973.

[6] F. Cariello and M. Tabor, "Similarity reduction from extended Painlevé expansion for nonintegrable evolution equations," *Physica D*, vol. 53, pp. 59-70, 1991.

[7] E. J. Parkes and B. R. Duffy, "An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations," *Comput. Phys. Commun.*, vol. 98, pp. 288-300, 1996.

[8] E. G. Fan, "Extended tanh-function method and its applications to nonlinear equations," *Phys. Lett. A.*, vol. 277, pp. 212-218, 2000.

[9] Z. Y. Yan, "New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations," *Phys. Lett. A.*, vol. 292, pp. 100-106, 2001.

[10] S. K. Liu, Z. T. Fu, S. D. Liu, and Q. Zhao, "Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations," *Phys. Lett. A.*, vol. 289, pp. 69-74, 2001.

[11] Z. Y. Yan, "Abundant families of Jacobi elliptic functions of the (2 + 1)-dimensional integrable Davey-Stewartson-type equation via a new method," *Chaos Solitons Fractals*, vol. 18, pp. 299-30, 2003.

[12] M. L. Wang and Y. B. Zhou, "The periodic wave equations for the Klein-Gordon-Schordinger equations," *Phys. Lett. A.*, vol. 318, pp. 84-92, 2003.

[13] Y. B. Zhou, M. L. Wang, and T. D. Miao, "The periodic wave solutions and solitary wave solutions for a class of nonlinear partial differential equations," *Phys. Lett. A.*, vol. 323, pp. 77- 88, 2004.

[14] M. L. Wang and X. Z. Li, "Extended F-expansion and periodic wave solutions for the generalized Zakharov equations," *Phys. Lett. A.*, vol. 343, pp. 48-54, 2005.

[15] M. L. Wang and X. Z. Li, "Applications of F-expansion to periodic wave solutions for a new hamiltonian amplitude equation," *Chaos, Solitons and Fractals*, vol. 24 pp. 1257-1268, 2005.

[16] J. L. Zhang, M. L. Wang, and X. Z. Li, "The subsidiary ordinary differential equations and the exact solutions of the higher order dispersive nonlinear Schrödinger equation," *Phys. Lett. A.*, vol. 357, pp. 188-195, 2006.

[17] M. L. Wang, X. Z. Li, and J. L. Zhang, "Various exact solutions of nonlinear Schrödinger equation with two nonlinear terms," *Chaos Solitons Fractals*, vol. 31, pp. 594-601, 2007.

[18] X. Z. Li and M. L. Wang, "A sub-ODE method for finding exact solutions of a generalized KdV–mKdV equation with high-order nonlinear terms," *Phys. Lett. A.*, vol. 361, pp. 115- 118, 2007.

[19] M. L. Wang, X. Z. Li, and J. L. Zhang, "Sub-ODE method and solitary wave solutions for higher order nonlinear Schrödinger equation," *Phys. Lett. A.*, vol. 363, pp. 96-101, 2007.

[20] M. L. Wang, "Solitary wave solutions for variant Boussinesq equations," *Phys. Lett. A.*, vol. 199, pp. 169-172, 1995.

[21] M. L. Wang, "Exact solutions for a compound KdV-burgers equation," *Phys. Lett. A.*, vol. 213, pp. 279-287, 1996.

[22] M. L. Wang, Y. B. Zhou, and Z. B. Li, "Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics," *Phys. Lett. A.*, vol. 216, pp. 67-75, 1996.

[23] A. M. Wazwaz, "Distinct variants of the KdV equation with compact and noncompact structures," *Appl. Math. Comput.*, vol. 150, pp. 365-377, 2004.

[24] A. M. Wazwaz, "Variants of the generalized KdV equation with compact and noncompact structures," *Comput. Math. Appl.*, vol. 47, pp. 583-591, 2004.

[25] X. Feng, "Exploratory approach to explicit solution of nonlinear evolutions equations," *Int. J. Theo. Phys.*, vol. 39, pp. 207-222, 2000.

[26] J. L. Hu, "A new method for finding exact traveling wave solutions to nonlinear partial differential equations," *Phys. Lett. A.*, vol. 286, pp. 175-179, 2001.

[27] J. L. Hu, "Explicit solutions to three nonlinear physical models," *Phys. Lett. A.*, vol. 287, pp. 81-89, 2001.

[28] J. L. Hu, "A new method of exact traveling wave solution for coupled nonlinear differential equations," *Phys. Lett. A.*, vol. 322, pp. 211-216, 2004.

[29] J. H. He and X. H. Wu, "Exp-function method for nonlinear wave equations," *Chaos, Solitons and Fractals*, vol. 30, pp. 700-708, 2006.

[30] M. Wang, X. Li, and J. Zhang, "The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Phys. Lett. A.*, vol. 372, pp. 417–423, 2008.

[31] M. Wang, X. Li, and J. Zhang, "Application of (G'/G)-expansion method to travelling wave solutions of the broer-kaup and the approximate long water wave equations," *Appl. Math. Comput.*, vol. 206, pp. 321–326, 2008.

[32] A. Ismail, "Exact and explicit solutions to some nonlinear evolution equations by utilizing the (G'/G)-expansion method," *Appl. Math. Comput.*, vol. 215, pp. 857-863, 2009.

[33] J. Zhang, X. Wei, and Y. Lu, "A generalized -(G'/G) expansion method and its applications," *Phys. Lett. A.*, vol. 372, pp. 3653–3658, 2008.

[34] S. Zhang, W. Wang, and J. L. Tong, "A generalized (G'/G)-expansion method and its application to the (2+1)-dimensional broer-kaup equations," *Appl. Math. Comput.*, vol. 209, pp. 399–404, 2009.

[35] S. Zhang, J. L. Tong, and W. Wang, "A generalized (G'/G)-expansion method for the mKdV equation with variable coefficients," *Phys. Lett. A.*, vol. 372, pp. 2254–2257, 2008.

[36] S. Zhong, L. Dong, J. Ba, and Y. Sun, "The (G'/G)-expansion method for nonlinear differential-difference equations," *Phys. Lett. A.*, vol. 373, pp. 905–910, 2009.

[37] Z. Yu-Bin and L. Chao, "Application of modified (G'/G)-expansion method to traveling wave solutions for Whitham broer kaup-like equations," *Commu. Theor. Phys.*, vol. 51, pp. 664–670, 2009.

[38] G. B. Whitham, *Linear and nonlinear waves.* New York: Academic Press, 1973.

[39] Z. T. Fu, S. D. Liu, and S. K. Liu, "New solutions to mKdV equation," *Phys. Lett. A.*, vol. 326, pp. 364-374, 2004.

[40] V. V. Gudkov, "A family of exact travelling wave solutions to nonlinear evolution and wave equations," *J. Math. Phys.*, vol. 38, pp. 4794-4803, 1997.

[41] W. X. Ma and B. Fuchssteiner, "Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation," *Int. J. Nonlinear Mech.*, vol. 31, pp. 329-338, 1996.

[42] M. Yoshimasa, "Reduction of dispersionless coupled Korteweg–De Vries equations to the Euler–Darboux equation," *J. Math. Phys.*, vol. 42, pp. 1744-1760, 2001.