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A SIMPLE CRITERION FOR THE NON-EXISTENCE OF LIMIT CYCLES OF A LIÉNARD SYSTEM

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ABSTRACT

In this paper, as an application in our results, the non-existence of limit cycles for the Liénard system $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$ *with* $F(x) = (x^2 - x)e^{-x}$ $(x \geq -1)$ *and* $5(x^2 + x)e^{x+2} + 2e$ $(x \leq -1)$, $g(x) = x$ *is discussed by the simple criterion. Graef [1] in 1971 has studied the uniformly boundedness of the solution orbits under the condition (C1) and further proved the existence of limit cycles under the conditions (C1) and (C2) . Recently, Cioni and Villari [2] in 2015 gave the same result as in Graef [1] under the conditions (C1) and (C3) includes (C2). Our aim is to discuss on the case of which (C1) is satisfied, but (C3) is not satisfied. As the result, we shall give the simple criterion for the non-existence of limit cycles for a Liénard system with these conditions.*

Keywords: Liénard system, Limit cycles, Globa l unstability, Non-existence, Homoclinic orbits, Uniformly boundedness.

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1. INTRODUCTION

In this paper, we consider a classical Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x) \tag{L}$$

where $F(x)$ and $g(x)$ are continuous on an open interval which contains the origin, satisfy smoothness conditions for the uniqueness of solutions of initial value problems.

The qualitative property of the closed orbits of this system has studied by many mathematicians, physicists, economists and engineers and so on. Thus, the results play an important role to resolve the scientific phenomenon.

Our purpose is to discuss on the non-existence of limit cycles and Global unstability of solution orbits under the case of which the recent condition isn't satisfied (see (C3) below). Our results can apply to the interesting example as is shown in Section 4. We shall see that it is proved by a simple criterion of the non-existence of limit cycles.

Throughout, we assume the following standard conditions for the limit cycle of system (L)

(H1) $g(x)/x > 0$, (H2) $g(x)F(x) < 0$ for $|x| < \varepsilon$ and ε is small are satisfied.

Graef [1] in 1971 has studied the uniformly boundedness of the solution orbits under the condition

(C1) $F(x) \pm G(x) \to \pm\infty$ $(x \to \pm\infty)$, $G(x) = \int_0^x g(\tau)d\tau$.

Further, he proved the existence of limit cycles under the conditions (C1) and

(C2) $\exists k > 0$ such that $F(x) \geq A(x \geq k), F(x) \leq -A(x \leq -k)$ for some fixed constant A.

Recently, Cioni and Villari [2] in 2015 gave the following

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Proposition System (L) with the conditions (C1) and

(C3) $\exists K_1 > \exists K_2$ such that $F(x) \geq K_1(x > \beta > 0), F(x) \leq K_2(x < \alpha < 0)$ has at least one limit cycles.

This is the same result as in Graef [1] but remark that (C3) is an improvement of (C2).

We are interesting in the case of which (C1) is satisfied, but (C3) is not satisfied. As the result, we shall give a simple criterion for the non-existence of limit cycles of system (L) with these conditions. The results shall be stated in the next section and proved in Section 3. In Section 4, we shall present a phase portrait of the concrete example for system (L).

2. RESULTS

From (H2), we divide the results to four cases by whether the curve $y=F(x)$ intersect the x-axis.

(i) $\nexists a_1$, $\nexists a_2$, (ii) $\exists a_2 > 0$, (iii) $\exists a_1 < 0$, (iV) $\exists a_1 < 0 < \exists a_2$ where a_i (i=1,2) are solutions of the equation $F(x) = 0$.

We shall give our results for the non-existence of limit cycles. The following plays an important role for our purpose.

2.1. Lemma

(Gasull and Guillamon [3])

If the plane curve ($F(x)$, $G(x)$) has no intersecting points with itself for all $x \in (a, b)$, system (L) has no limit cycles contained in the domain

$$D_1 = \{(x, y) \mid a < x < b, y \in \boldsymbol{R}\} \quad \text{or} \quad D_2 = \{(x, y) \mid x < b, y \in \boldsymbol{R}\} \quad \text{or} \quad D_3 = \{(x, y) \mid a < x, y \in \boldsymbol{R}\}.$$

2.2. Theorem

(Case (i))

If the curve $y=F(x)$ doesn't intersect the x-axis except the origin, system (L) has no limit cycles and the equilibrium point (0,0) is globally unstable.

It is clear from the above lemma or the Lemma 3.1 in Hayashi [4].

Assume that there exists $\exists K_1$ *and* $\exists K_2$ such that $F(x) \geq K_1(x > \beta > 0), F(x) \leq K_2(x < \alpha < 0)$. The following results are the case of which the condition [C3] isn't satisfied.

2.3. Theorem

(Case (ii))

Let α_1 be a negative number such that $G(a_2) = G(\alpha_1)$. If system (L) satisfies the conditions (C1) and (C4) $(0 <)\ K_1 < M = min(F(\alpha_1),\ K_2)$, then the system has no limit cycles.

2.4. Theorem (Case (iii))

Let α_2 be a positive number such that $G(a_1) = G(\alpha_2)$. If system (L) satisfies the conditions (C1) and (C5) $K_2 < N = max(F(\alpha_2),\ K_1)\ (< 0)$, then the system has no limit cycles.

2.5. Corollary

Under the conditions in Theorem 2.3 or Theorem 2.4 the equilibrium point (0,0) of system (L) is globally unstable.

Remark. In the case (iv), we see from Cioni and Villari [2] or Graef [1] that system (L) has at least one limit cycles.

3. PROOFS

First, we shall prove Theorem 2.3.

For all u_1 and u_2 such that $G(u_1) = G(u_2)$, $\alpha_1 \leq u_1 < 0$ and $0 < u_2 \leq a_2$, we have $F(u_1) > 0$ and $F(u_2) < 0$. Thus, we get $F(u_1) - F(u_2) > 0$. On the other hand, for all u_1 and u_2 such that $G(u_1) = G(u_2)$, $u_1 < \alpha_1$ and $a_2 < u_2$, we have from the condition (C4) that $F(u_1) - F(u_2) > M - K_1 > 0$. These facts means that the plane curve $(F(x), G(x))$ has no intersecting points with itself. Therefore, we conclude from Lemma 2.1 that system (L) has no limit cycles. The proof of Theorem 2.4 is given by the same discussion as Theorem 2.3.

Next, we prove Corollary 2.5.

Under the conditions in Theorem 2.3 or Theorem 2.4 system (L) has no limit cycles. Further, from the condition (H2) system has no homoclinic orbits(see Hayashi [5]) and the equilibrium point (0,0) is unstable. Thus, the origin is globally unstable.

4. AN EXAMPLE

We shall apply our results to the system below and present a phase portrait of the system.

Example. Consider system (L) in the form

$$F(x) = \begin{cases} (x^2 - x)e^{-x} & \text{for } x \geq -1 \\ 5(x^2 + x)e^{x+2} + 2e & \text{for } x \leq -1 \end{cases}, \qquad g(x) = x.$$

We can take from $G(x) = x^2/2$ that $a_2 = 1, \alpha_1 = -1$, $K_1 = 4.3 / e^{2.6} < K_2 = 2e$ and $F(-1) = 2e$. Since $K_1 < M = 2e$, the system satisfies all conditions of Theorem 2.3. Thus, we see that this system has no limit cycles as is shown in the Figure 1 below.

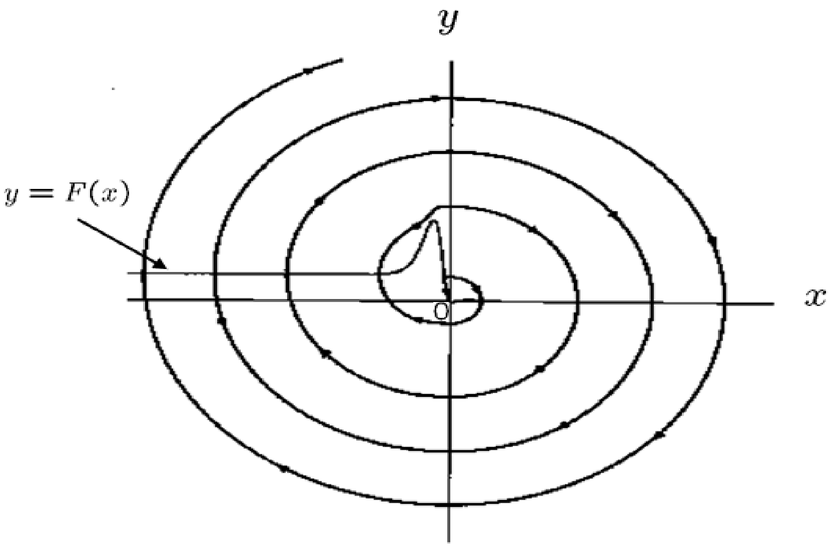

Figure-1. (The phase portrait of the system for the above example)

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