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THE HARTMAN-WINTNER LAW OF THE ITERATED LOGARITHM FOR NONCOMMUTATIVE MARTINGALES

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ABSTRACT

In this study, we prove one of the fundamental strong laws of classical probability theory, the Hartman-Wintner's law of the iterated logarithm for non-commutative martingale using a simple exponential inequality.

Keywords: Hartman-Wintner's law, Kolmogorov's law, Non-commutative martingale, Law of the iterated logarithm (LIL).

MSC: 60B05, 60F10, 46L53, 60F15.

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Contribution/ Originality

This study contributes in the existing literature by proving the Hartman-Wintner's law of the iterated logarithm for non-commutative martingale using a simple exponential inequality as a counterpart of the Kolmogorov's law of the law iterated logarithm.

1. INTRODUCTION

The early contributions in the law of iterated logarithm for independent increments were made by Khintchine, Kolmogorov, Hartman-Wintner and others (see Bauer [1]). It was Stout that generalized Kolmogorov and Hartman-Wintner's results to martingales (see Stout [2] and Stout [3]). Kuelbs, Ledoux, Talagrand, Pisier and others extended LIL to independent sums in Banach spaces see [4]. However, it seems that the LIL in non-commutative (quantum) probability theory for Hartman-Wintner's version has not received much attention.

There are many reasons why non commutative martingales are of interest since classical mathematical finance theory is a well developed discipline of applied mathematics which has numerous applications in financial markets. There is a great interest in generalizing this theory to the domain of quantum probabilities since the theory has its foundation on probability. It has been shown currently that the quantum version of financial markets is better suitable to real world financial markets rather than the classical one, because the quantum binomial model does not pose ambiguity which appears in the classical model of the binomial market [5].

Our interest in this paper is to prove the Hartman-Wintner's law of the iterated logarithm for non-commutative martingale using a simple exponential inequality as a counterpart of the Kolmogorov's law of the law iterated logarithm.

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1.1. Hartman – Wintners Law of the Iterated Logarithm

Let $\{X_i, i \geq 1\}$ be an independent identically distributed random variables with $EX_1 = 0$ and $EX_1^2 = 1$, taking $S_n = \sum_{i=1}^{n} X_i$and $S_n^2 = Var\,(S_n) = \sum_{i=1}^{n} E(X_i^2)$ (note that E denotes the expectation and var denotes the variance). Hartman – Wintner's law of the iterated logarithm (LIL) states that

$$\lim_{n\to\infty} sup \sum_{i=1}^{n} \frac{X_i}{\sqrt{2n \log logn}} = \sigma\,, \quad \text{a. s.}$$

The non-commutative probability space is (M, τ). Here M is finite von Neumann algebra and τ a faithful tracial state, i.e.

$$\tau(xy) = \tau(yx) \text{ for } x, y \in M.$$

Given $1 \leq p < \infty$, define $\|x\|_p = [\tau(|x|^P)]^{\frac{1}{P}}$ and $\|x\|_\infty = \|x\|$ for $x \in M$. Here $\|\cdot\|$ denotes the operator norm.

The commutative L_P space $L_P(M, \tau)$ or $L_P(M)$ for short is the completion of M with respect to $\|\cdot\|_P$. $\tau -$measurable operators affiliated to $(M, \tau)$are also called non-commutative random variables.

Let $(M_k)_{k=1,2\,\dots} \subset M$ be a filtration of von Neumann subalgebras with conditional expectation. If $E_k\colon M \to M_k$. Then $E_k(1) = 1$ and $E_k(axb) = a\,E_k(x)b$ for $a, b \in M_k$ and $x \in M$. It is widely known that E_k extends to contractions on $L_P(M, \tau)$for $p \geq 1$ [6].

In Konwerska [7] x_n of $\tau -$ measurable operators is said to be almost uniformly bounded by a constant $k \geq 0$, denoted by

$$\lim_{n\to\infty} sup\, x_n \leq k,$$

if for any $\in > 0$ and any $\delta > 0$, there exists a projection e with $\tau(I - e) < \in$ such that

$$\lim_{n\to\infty} sup\, \|x_n e\| < k + \delta\,. \qquad (1)$$

(x_n) is said to be bilaterally almost uniformly bounded by a constant $k \geq 0$ denoted by

$$\lim_{n\to\infty} sup\, x_n \leq k \qquad b.a.u$$

if (1) is replaced by

$$\lim_{n\to\infty} sup\, \|e\, x_n e\| < k + \delta$$

obviously

$$\lim_{n\to\infty} sup\, x_n \leq k \qquad a.u$$

implies

$$\lim_{n\to\infty} sup\, x_n \leq k \qquad b.a.u.$$

For a $\tau -$ measureable operator x and $t > 0$, the generalized singular numbers in Fack and Kosaki [8] are defined by

$$U_t(x) = inf\{s > 0\colon \tau\big(1_{(s,\infty)}(|x|) \leq t\big\}.$$

A sequence of operators (x_i) is uniformly bounded in distribution by an operators y if there exist $k > 0$ such that (see Konwerska [7])

$$sup_i u_t(x_i) \leq k u_t/k(y) \quad \forall\ t > 0.$$

Let (x_n) be a sequence of mean zero self adjoint independent random variables Konkwerska [9] proved that if (x_n) is uniformly bounded in distribution by a random variable y such that $\tau(|y|^2) = \sigma^2 < \infty$, then

$$\lim\sup_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{\sqrt{2nloglogn}} < c\sigma\,, \qquad b.a.u.$$

If (x_n) is $i.i.d$, then (x_n) is uniformly bounded in distribution by x_i independent and identically distributed random variables form the sequence of Hartman Wintner's LIL and for a sequence to be uniformly bounded in distribution the sequence should be almost identically distributed.

2. PRELIMINARIES

Lemma 1: Let X_i be independent random variables and $T_n = \sum_{i=1}^{n} X_i$.

Assume (i) $\frac{T_n}{a_n} \to 0$ (in probability)

(ii) for some $\propto > 1, \beta > 0,\ C > 0$ and for $n_o \in N$,

$P\{T_n/a_n > \beta\} \leq c\ exp\ \{-\propto LLn\}forn \geq n_o$. Then $P\{\lim\sup \mathrm{T_n}/a_\mathrm{n} < \beta\} = 1$

Lemma 2 : Let $\{x_i\}$ be independent, identically distributed random variables,$T_n = \sum_{i=1}^{n} X_i$.

Assume (i) $EX_i = 0, \quad Sup\ E|X_i|^2 < \infty$

(ii) $|X_i| < \tau(i/LL_i)^{\frac{1}{2}}$ a. s for all i and some $\tau > 0$.

Then if $a \geq (sup_i\ E\ [X_i]^2)^{\frac{1}{2}}\ \ \forall\ t > 0, n \in N$

$$P\{T_n/a_n > t\} \leq exp\left\{\left(-{}^{t}/_{a}\right)^2 \left(2 - exp\left(\sqrt{2}\ ta^{-2}\tau\right)\right) LL_n\right\}.$$

Proof: Since $e^x \leq 1 + x + (x^2/2)e^{|x|}$ for all real x and for $j \leq n$

$$|X_i/a_n| \leq \tau(i/LLi)^{\frac{1}{2}}/(2nLLn)^{\frac{1}{2}} \leq \tau(n/LLn)^{\frac{1}{2}}/(2nLLn)^{\frac{1}{2}} \leq \tau/\sqrt{2}LLn$$

we have $\forall\ \lambda > 0$

$$\exp(\lambda X_i/a_n) \leq 1 + \frac{\lambda X_i}{a_n} + \frac{\lambda^2 X_i^2}{2a_n^2}\ exp\ \left(\lambda\tau/\sqrt{2}\ LLn\right).$$

Taking expectations for $i \leq n$

$$\mathrm{E}\exp(\lambda X_i/a_n) \leq 1 + \frac{\lambda^2 X_i^2}{2a_n^2}\exp\ \left(\lambda\tau/\sqrt{2}LLn\right) \leq \exp\left\{\frac{\lambda^2 a^2}{4nLLn}\ exp\ \left(\lambda\tau/\sqrt{2}LLn\right)\right\}$$

and by independence

$$E\ exp\ (\lambda T_n/a_n) = \prod_{i=1}^{n} E\exp(\lambda X_i/a_n) \leq \exp\left\{\frac{\lambda^2 a^2}{4nLLn}\ exp\ \left(\lambda\tau/\sqrt{2}LLn\right)\right\} \forall\ \lambda > 0, t > 0.$$

By Markov's inequality,

$$P\{T_n/a_n > t\} \leq exp(-\lambda\ t)\ E\ exp(\lambda T_n/a_n) \leq \exp\left\{-\lambda t + \frac{\lambda^2 a^2}{4LLn}\ exp\ \left(\lambda\tau/\sqrt{2}LLn\right)\right\} \qquad (2)$$

for fixed t, we set $\lambda = a^{-2}2tLLn$ in (2) and this yields the stated inequality.

Lemma 3[10]: Let $\{X_i\}$ be independent identically distributed random variables with $E|X_1|^2 < \infty$. Let $\tau > 0$, and define $Z_i = X_i I\left\{|X_i| > \tau\ (i/LL_i)^{\frac{1}{2}}\right\}$and $U_n = \sum_{i=1}^{n} Z_i$.

Then

$$P\left\{\lim_{n\to\infty} |U_n/a_n| = 0\right\} = 1.$$

2.1. Non Commutative L_p Spaces

The vector valued non-commutative L_p spaces for $I \leq p \leq \infty$. Let (x_n) be a sequence in $L_p(\aleph)$ and define

$$\|(x_n)\|_{L_{P(l\infty)}} = inf\left\{\|(a)\|_{2_P}\|b\|_{2_P}: x_n = ay_nb\|y_n\|_\infty \leq 1\right\}.$$

Then $L_p(l_\infty)$ is defined to be the closure of all sequences with $\|(x_n)\|_{L_{P(l\infty)}} < \infty$. It was shown in Junge and Xu [11] that if every x_n is self adjoint then

$$\|(x_n)\|_{L_{P(l\infty)}} = inf\left\{\|(a)\|_p: a \in L_p(\aleph), a > 0, -a \leq x_n \leq a\ \ \forall n \in N\right\}.$$

The space $L_p(l_\infty)$ with norm was introduced in [6] by Junge and X_u

$$\|(x_i)i \in I\|_{L_{P(l\infty)}} = inf\left\{\|(a)\|_p: a \in L_p(\aleph), a \geq 0, -a \leq x_i^* x_i \leq a \quad \forall i \in I\right\}$$

$$= inf\{\|b\|_p : x_i = y_i b_i \|y\|_\infty \leq \quad \forall i \in I\}$$

Theorem 2 [12]**:** Let $4 \leq p \leq \infty$. Then, for any $x \in L_p(\aleph)$ there exists $b \in L_p(\aleph)$ and a sequence of contractions $(y_n) \subset \aleph$ such that $\|b\|_p \leq 2^{2/p}\|x\|_p$ and $E_n x = y_n b \quad \forall n \geq 0$

This is the non-commutative asymmetric version of Doob's maximal inequality proved by Junge [12]

Proof: From Junge [12] corollary 4.6 setting $r = p \geq 4$ and $q = \infty$, we find $E_n x = az_n b$for $a, Z_n \in \aleph$ and $b \in L_p(\aleph)$. Let $y_n = az_n/\|az_n\| \in \aleph$ and $b^1 = \|az_n\| b \in L_p(\aleph)$. Then (y_n) is a sequence of contractions $E_n x = Y_n b^1$ and

$$\|b^1\|_p \leq \|a\|_\infty \|b\|_p \sup_n \|z_n\|_\infty < c(p,q,r)\|x\|_p,$$

where $c(p,q,x) \leq c_q^{\frac{1}{2}}/(q-2)c_{r/(r-2)}^{\frac{1}{2}} = c_1^{\frac{1}{2}} c_p/(p-2)$ and c_p is the constant in the dual Doob's inequality.

Note that $l \leq \frac{p}{(p-2)} \leq 2$ and by Lemma 3.1 and 3.2 of Junge and Xu [11] we find that

$$C_p \leq 2^{2(p-1)/p},\ 1 \leq p \ \leq 2.$$

It follows that $c(p,q,r) \leq 2^{2/p}$.

Assuming $(x_i)_{m \leq i \leq n}$ is a martingale in $L_p(\aleph)$ there exists $b \in L_p(\aleph)$ and contractions $(y_i)_{m \leq i \leq n} \subset \aleph$ such that $x_i = y_i b$ for $m \leq i \leq n$ and

$$\|b\|_p \leq 2^{2/p}\left\|x_p\right\|_{p'}, p \geq 4.$$

It follows that $\|(x_i)_{m \leq i \leq n}\|_{L_{p(l\infty)}} \leq 2^{\frac{2}{p}}\|x_n\|_p$.

This is the form in which Doob's inequality will be applied.

Lemma 4: Let (x_n) be a self adjoint martingale with respect to the filtration $(\aleph_k, E_k)$ and $d_k = x_k - x_{k-1}$ be the associated martingale difference such that

(i) $\tau(x_k) = x_\circ = 0$

(ii) $\|d_k\| \leq M$

(iii) $\sum_{k=1}^n E_{k-1}(d_k^2) \leq D^2$.

Then $\tau\left(e^{\lambda} x_n\right) \leq \exp[(1+\varepsilon)\lambda^2 D^2]\ \forall\, \varepsilon \in (0,1]$and all $\lambda \in \left[0, \sqrt{\varepsilon}/(M + M\varepsilon)\right.$.

Remark: For a self adjoint sequence $(x_i)_{i \in I}$ of random variables, the column version of tail probability is given as in [7].

$$p_c(sup\|x_i\| > t) = inf\{S > 0 : \exists \text{ a projection c with } \tau(1-c) < s\}$$
$$\text{and } \|x_i c\|_\infty \leq t\ \forall\, i \in I\}\ for\ t > 0$$

It is immediate that

$$p_c\left(\sup_{i \in I}\|x_i\| > t\right) \leq p_c\left(\sup_{i \in I}\|x_i\| > r\right) \text{ for } t \geq r \qquad (3)$$

and that if $a_i \geq 1$ for $i \in I$, then

$$p_c\left(\sup_{i \in I}\|x_i\| > t\right) \leq p_c\left(\sup_{i \in I}\|a_i x_i\| > t\right) \qquad (4)$$

From this we state the following

Lemma 5:(Non-commutative Borel – Cantelli Lemma)

Let $U_n I_n = \{n \in \mathbb{N} : n \geq n_\circ\}$ for some $n_\circ \in \mathbb{N}$ and(Z_n) be a sequence of self adjoint random variables. If for any $\delta > 0$, $\sum P_c(sup\|Z_m\| > \gamma + \delta) < \infty$. Then

$$\lim_{n\to\infty} sup\, Z_n \underset{a.u}{\leq} \gamma. \qquad (5)$$

Lemma 6:(Non-commutative Chebyshev inequality)

Let $(x_i)_{i \in I}$ be a self-adjoint sequence of random variables. For $t > 0$ and $1 \leq p < \infty$

$$p_c(sup\|x_n\| > t) \leq t^p \|x_n\|^p L_{p(l\infty)}.$$

Our proof of LIL for martingales follows the exponential inequality proved in Junge and Zeng [13].

Hartman – Wintner's law of the iterated logarithm (LIL) states that

$$\lim_{n\to\infty} sup \sum_{i=1}^{n} \frac{X_i}{\sqrt{2nloglogn}} = \sigma \qquad a.s$$

We generalize this result to stationary ergodic martingale difference sequences.

Theorem 2: Let $(x_i, i \geq 1)$ be a stationary ergodic sequence with

$$E[X_i/X_1, X_2, \cdots X_{i-1}] = 0 \qquad a.s$$

$\forall\, i \geq 2$ and $E\, X_1^2 = 1$

Then

$$\lim_{n\to\infty} sup \frac{\sum_{i=1}^{n} X_i}{\sqrt{2nloglogn}} = 1 \qquad a.s$$

According to Stout [14] if $(X_i, \mathcal{F}_i, \geq 1)$ is a martingale difference sequence with

$$S_n^2 = \sum_{i=1}^{n} E[X_1^2/\mathcal{F}_{i-1}] \to \infty, \qquad a.s$$

$U_n = (2\, loglogS_n^2)^{\frac{1}{2}}$, $\mathcal{F}_{i-1}$measurable random variables $L_i \to 0$ $a.s$, and $|X_i| \leq L_i S_i/_{u_i}$ a.s

$\forall = i \geq 1.$

Then

$$\lim_{n\to\infty} \sup \sum_{i=1}^{n} \frac{X_i}{(S_n U_n)} = 1 \qquad a.s$$

Note:

$$\sum_{i=1}^{n} E\left[\frac{(X_1')^2}{\mathcal{F}_{i-1}}\right]/n \to 1 \qquad a.s \quad \ldots \quad (6)$$

and hence

$$\sum_{i=1}^{n} E\,[(X_1')^2/\mathcal{F}_{i-1}] \to \infty \qquad a.s \quad \ldots \quad (7)$$

x_1'satisfies the hypotheses of this theorem with $L_1 = 2k_i u_i (i/loglogi)^{\frac{1}{2}}/S_i$ since

$$|X_1'| \leq 2\, k_i (i/loglog\, i)^{\frac{1}{2}}. \qquad \text{a.s}$$

Thus using (4),

$$\lim sup \sum_{i=1}^{n} \frac{X_1'}{(2nloglogn)^{\frac{1}{2}}} = 1.$$

2.1. Main Result

Theorem 1: Let $X = x_o, x_1, x_2, \ldots$ be a self-adjoint martingale in $(\aleph, \tau)$. Suppose $S_n^2 \to \infty$ and $\|d_n\|_\infty \leq \propto_n s_n/u_n$ for some sequence $(\propto_i)$ of positive number such that $\propto_n \to 0$ as $n \to \infty$, then the Hartman-Wintner's law of the iterated logarithm for non-commutative martingale is given by

$$\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2n \log \log n}} \leq \sigma \qquad \text{a.s}$$

This is the first result of the LIL for non-commutative martingales. Since there is no LIL lower bound in the general non-commutative theory one can only expect an upper bound for LIL in the general non-commutative setting.

2.2. Proof

Let $\omega \in (1,2)$ be a constant to be determined, using the stopping rule see [2]. We define $k_o = 0$ for $n \geq 1$ and we use $S(k_i) = S_{k_i}$

$$k_o = inf\left(j \in \mathbb{N}: S_{j+1}^2 \geq \omega^{2n}\right),$$

then $S_{k_n+1}^2 \geq \omega^{2n}$ and $S_{k_n+1}^2 < \omega^{2n}$.

Note that given $\varepsilon > 0$ there exists $N_1(\varepsilon) > 0$ such that for $n > N_1(\varepsilon)$

$S_{k_n+1}^2 \ U_{k_n+1}^2 / \ (S(k_{n+1})^2 \ U(k_{n+1})^2 \geq \omega^{-2} InIn\, \omega^{2n} / InIn\, \omega^{2(n+1)} \geq (1-\varepsilon)^2 \omega^{-2}$.

Then $S_m U_m \geq (1-\epsilon)\omega^{-1} S(k_{n+1})\, U(k_{n+1})$ for $k_n < m \leq k_{n+1}$ for any $\delta > 0$ we can find $\delta\, \varepsilon > 0$ and $\omega \in (1,2)$ such that $1-\delta > \omega\,(1+\delta)\,(1-\epsilon)^{-1}$. Fix $\beta > 0$ to be determined.

Using relations (3) and (4) we have for $n > N_i(\varepsilon)$

$$P_c\left(\begin{matrix} sup \\ k_n < m < k_{n+1}\end{matrix} \left\|\frac{x_n}{S_n\, U_n}\right\| > \beta(1+\delta)\right) \leq P_c\left(sup \left\|\frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}}\right\| > \lambda\beta(1+\delta)\right). \qquad (8)$$

By Lemma 6 and theorem 2 we have for $p \geq 4$

$$P_c\left(sup \left\|\frac{\lambda\dot{\tau}}{(2loglogn)^{\frac{1}{2}}}\right\| > \lambda\beta(1+\delta)\right) \leq \left(\lambda\beta(1+\delta)\right)^{-P} \left\|\frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}}\right\|_{L_p(\ell_\infty)}^P$$

$$\leq \left(\lambda\beta(1+\delta)\right)^{-P} \left(2^{2/P}\right)^P \left\|\frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}}\right\|_P^P.$$

Applying the elementary inequality

$|U|^P \leq \ P^p\, e^{-p}\, (e^u + e^{-u})$ together with functional calculus and Lemma 5 we have

$$\left\|\frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}}\right\|_P^P \leq P^p\, e^{-p}\tau \left\{exp\left(\frac{\lambda\tau}{(2loglog\, n)^{\frac{1}{2}}} + exp\left(\frac{\lambda\tau}{(2loglog\, n)^{\frac{1}{2}}}\right)\right)\right\},$$

where

$$exp\left(\frac{\lambda\tau}{(2loglog\, n)^{\frac{1}{2}}}\right) = \lambda\left(1 + \frac{\tau}{\sqrt{2LLn}} + \frac{\tau^2}{2LLn} + \cdots\right),$$

and

$$exp\left(\frac{-\lambda\tau}{(2LLn)^{\frac{1}{2}}}\right) = \lambda\left(1 - \frac{\tau}{\sqrt{2LLn}} + \frac{\tau^2}{2LLn} + \cdots\right)$$

$$\leq \left(\frac{P}{e}\right)^p \lambda\left(2 + \frac{\tau^2}{LLn}\right)$$

provided

$$0 \leq \lambda \leq \frac{\sqrt{\varepsilon}\, u(k_{n+1})^2}{(1+\varepsilon) \propto (k_{n+1})} \ and\ 0 < \varepsilon \leq 1.$$

Hence we obtain

$$P_c\left(sup \left\| \frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}} \right\| > \lambda\beta(1+\delta) \right) \leq\ 8\ \left(\frac{P}{\lambda\beta(1+\delta)\varepsilon}\right)^{p} exp\ \left(2 + \frac{\tau^2}{loglog\ n} - \ \beta(1+\delta)\lambda\right).$$

Not optimizing in p give $p = \lambda\beta(1+\delta)$and thus

$$P_c\left(sup \left\| \frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}} \right\| > \lambda\beta(1+\delta) \right) \leq 8\ exp\ \left(2 + \frac{\tau^2}{loglog\ n} - \ \beta(1+\delta)\lambda\right).$$

Put $\lambda = \ \beta(1+\delta)\ u(\ k_{n+1})^2/(2(1+\varepsilon))$.

Since $\propto_n \rightarrow 0$, for any $\varepsilon > 0$ there exists $N_2 > 0$ such that for $n\ > N_2$,

$$0\ <\ \alpha(k_{n+1})\ \leq \frac{2\sqrt{\varepsilon}}{\beta(1+\delta)},$$

which ensures that we can apply lemma 4.

This also implies $p\ \geq 4$ for large n it follows that

$$P_c\left(sup \left\| \frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}} \right\| > \lambda\beta(1+\delta) \right) \leq (lns\ (k_{n+1})^2)^{\ -\frac{\beta^{2(1+\delta)^2}}{4(1+\varepsilon)}}.$$

Notice that $S\ (k_{n+1})^2\ \geq S\ (k_n+1)^2\ \geq \omega^{2n}$.

Setting $\beta\ = 2$ in the beginning of the proof we have

$$P_c\left(sup \left\| \frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}} \right\| > \lambda\beta(1+\delta) \right) \leq |(2I\omega)\omega|^{\ -\frac{(1+\delta)^2}{1+\varepsilon}}.$$

By choosing ε small enough so that $(1+\delta)^2/(1+\varepsilon\ ) > 1$ we find that for

$$n_\circ = \max\{\mathbb{N}_1, \mathbb{N}_2\}$$

$$\sum_{n\geq n_\circ} P_c\ \left(sup \left\| \frac{\lambda\tau}{(2loglogn)^{\frac{1}{2}}} \right\| > \lambda\beta(1+\delta) \right) < \infty.$$

Then (7) and lemma (5) give the desired result.

Conclusion

By following the method in De Acosta [10] we have proved herein the Hartman-Wintner's law of the iterated logarithm for non-commutative martingale using a simple exponential inequality as a counterpart of the Kolmogorov's law of the law iterated logarithm [15].

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