



## ANALYSIS OF AN ECO-EPIDEMIOLOGICAL MODEL WITH DISEASE IN THE PREY AND PREDATOR

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### ABSTRACT

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We analyze and formulate an Eco-Epidemiological model with disease in the prey and predator, study the existence of the non-negative equilibria, obtain the sufficient conditions of locally asymptotical stability of the equilibria, then analyze the global stability of the positive equilibria.

#### Keywords

Eco-epidemiological model

Asymptotically stable

Equilibria

Liapunov function

Prey

Predator.

**Contribution/ Originality:** This study contributes in the existing literature of Eco-Epidemiological model. We get the conditions of local asymptotic and the existence of the boundary balance, and we proved the positive balance point is global asymptotical stability by constructing Liapunov function.

### 1. INTRODUCTION

Mathematical ecology and mathematical epidemiology are major fields of study. Since transmissible disease in ecological situation can't be ignored, it is very important from both the ecological and the mathematical points of view to study ecological systems subject to epidemiological factors. A number of studies have been performed in this field; However, all these papers available only discussed the disease spread in a species, seeing [1-3] deal with the disease is spread among the predator population only, but in literatures [4-7] the disease is spread among the preys population considered. In our common life, the disease may spread among the prey and the predator. On the basis of this, this paper deals with the prey-predator model with diseases in the prey and predator, and we suppose the predator with disease dose not capture on the preys, the susceptible predator capture both on the susceptible and on the infected prey, but the capture rate is different, which much closer to the actual situation. This paper consider the model as follows:

$$\begin{cases} \frac{dS_x}{dt} = rS_x \left[ 1 - \left( \frac{S_x + I_x}{K} \right) \right] - \beta_1 S_x I_x - k_1 S_x S_y \\ \frac{dI_x}{dt} = \beta_1 S_x I_x - k_2 I_x S_y - d_0 I_x \\ \frac{dS_y}{dt} = k_1 \theta S_x S_y - d_1 S_y - \beta_2 S_y I_y + k_2 \theta I_x S_y \\ \frac{dI_y}{dt} = \beta_2 S_y I_y - d_2 I_y \end{cases}$$

Where ,  $S_x, I_x, S_y, I_y$  be the densities of susceptible prey ,infected prey , susceptible predator and infected predator ,  $r$  stands for the intrinsic growth rate of the susceptible preys ,  $\beta_1, \beta_2$  represent the transmission rate of the susceptible prey and susceptible predator respectively,  $K$  be the environmental carrying capacity of the prey population ,  $k_1$  and  $k_2$  represent the capturing rate of susceptible predator on the susceptible prey and on the infected prey respectively,  $\theta$  be the conversion of the predator ,  $d_0$  and  $d_2$  be the death of infected prey on infected predator because of diseases ,  $d_1$  be the natural mortality of the susceptible prey.

All the parameters are assumed to be positive.

## 2. EQUILBRIA ANALYSIS

Let

$$\begin{cases} P(S_x, I_x, S_y, I_y) = rS_x \left[ 1 - \left( \frac{S_x + I_x}{K} \right) \right] - \beta_1 S_x I_x - k_1 S_x S_y = 0 \\ Q(S_x, I_x, S_y, I_y) = \beta_1 S_x I_x - k_2 I_x S_y - d_0 I_x = 0 \\ R(S_x, I_x, S_y, I_y) = k_1 \theta S_x S_y - d_1 S_y - \beta_2 S_y I_y + k_2 \theta I_x S_y = 0 \\ M(S_x, I_x, S_y, I_y) = \beta_2 S_y I_y - d_2 I_y = 0 \end{cases}$$

**Case I :** It is obvious that the system has non-negative equilibria point  $E_0(0,0,0,0)$ , and  $E_1(K,0,0,0)$ .

**Case II:**  $S_y = 0$ , by

$$\begin{cases} S_x r \left( 1 - \frac{S_x + I_x}{K} \right) - \beta_1 S_x I_x = 0 \\ I_x (\beta_1 S_x - d_0) = 0 \\ -d_2 I_y = 0 \end{cases}$$

We obtain:  $I_y = 0, S_x = \frac{d_0}{\beta_1} = S_{x2}, I_x = \frac{r(\beta_1 K - d_0)}{\beta_1(\beta_1 K + r)} = I_{x2}$ .  $E_2(S_x, I_x, 0, 0)$  is non-negative equilibria point

when  $\beta_1 K > d_0$ .

**Case III:**  $I_y = 0$ , by

$$\begin{cases} rS_x(1 - \frac{S_x + I_x}{K}) - \beta S_x S_y = 0 \\ I_x(\beta_1 S_x - k_2 S_y - d_0) = 0 \\ S_y(k_1 \theta S_x - d_1 + k_2 \theta I_x) = 0 \end{cases}$$

We obtain

$$S_y = \frac{[\beta_1 \theta K (\beta_1 d_0 + k_2 r) + \beta_1 r d_1] - d_0 \theta (K \beta_1^2 + 2k_2 r)}{k_2 \theta (K \beta_1^2 + 2k_2 r)},$$

$$S_x = \frac{\theta K (\beta_1 d_0 + k_2 r)}{K \theta \beta_1^2 + 2k_2 \theta r} = S_{x3}, I_x = \frac{d_1 (K \beta_1^2 + 2k_2 r - k_1 r) - \theta K k_1 (\beta_1 d_0 + k_2 r)}{k_2 (K \theta \beta_1^2 + 2k_2 \theta r)} = I_{x3},$$

We obtain non-negative equilibria point  $E_3(S_{x3}, I_{x3}, S_{y3}, 0)$  when

$$d_1 (K \beta_1^2 + 2k_2 r - k_1 r) > \theta K k_1 (\beta_1 d_0 + k_2 r), \beta_1 \theta K (\beta_1 d_0 + k_2 r) + \beta_1 r d_1 > d_0 \theta (K \beta_1^2 + 2k_2 r)$$

**Case IV :**  $S_x \neq 0, I_x \neq 0, S_y \neq 0, I_y \neq 0$ , by

$$\begin{cases} r \left( 1 - \frac{S_x + I_x}{K} \right) - \beta_1 I_x - k_1 S_y = 0 \\ \beta_1 S_x - k_2 S_y - d_0 = 0 \\ k_1 \theta S_x - d_1 - \beta_2 I_y + k_2 \theta I_x = 0 \\ \beta_2 S_y - d_2 = 0 \end{cases}$$

We get

$$S_x = \frac{kd_2 + \beta_2 d_0}{\beta_1 \beta_2} = S_x^*, I_x = \frac{r(K \beta_1 \beta_2 - d_0 \beta_2 - k_2 d_2) - k_1 \beta_1 d_2}{(r + \beta_1) \beta_1 \beta_2} = I_x^*,$$

$$S_y = \frac{d_2}{\beta_2} = S_y^*, I_y = \frac{k_1 \theta (d_0 \beta_2 + d_0 \beta_1 \beta_2 + k_2 d_2 r) + k_2 \theta r (K \beta_1 \beta_2 - d_0 \beta_2 - k_2 d_2)}{\beta_1 \beta_1^2 (r + \beta_1)} = I_y^*.$$

We obtain non-negative equilibria point  $E_4(S_x^*, I_x^*, S_y^*, I_y^*)$  when  $r(K \beta_1 \beta_2 - d_0 \beta_2 - k_2 d_2) > k_1 \beta_1 d_2$

### 3. STABLE ANALYSIS

The Jacobi matrix of the system is

$$J = \begin{vmatrix} r - \frac{2r}{K}S_x - (\frac{r}{K} + \beta_1)I_x - k_1S_y & (-\frac{r}{K} - \beta_1)S_x & -k_1S_x & 0 \\ \beta_1I_x & \beta_1S_x - k_2S_y - d_0 & -k_2I_x & 0 \\ k_1\theta S_y & k_2\theta S_y & k_1\theta S_x - d_1 - \beta_2I_y + k_2\theta I_x & -\beta_2S_y \\ 0 & 0 & \beta_2I_y & \beta_2S_y - d_2 \end{vmatrix}$$

**Case I :** For  $E_0(0,0,0,0)$  the characteristic equation is

$$(\lambda - r)(\lambda + d_0)(\lambda + d_1)(\lambda + d_2) = 0.$$

We can get the characteristic root as follows:

$$\lambda_1 = r > 0, \lambda_2 = -d_0 < 0, \lambda_3 = -d_1 < 0, \lambda_4 = -d_2 < 0$$

Therefore  $E_0(0,0,0,0)$  is a Saddle point.

**Case II :** For  $E_1(K,0,0,0)$  the characteristic equation is

$$(\lambda - r)(\lambda - \beta_1K + d_0)(\lambda + d_1)(\lambda + d_2) = 0.$$

Root of this equation are  $\lambda_1 = -r < 0, \lambda_2 = \beta_1K - d_0, \lambda_3 = -d_1 < 0, \lambda_4 = -d_2 < 0$ . Here  $\lambda_2 < 0$  when

$\beta_1K < d_0$  Hence  $E_1(K,0,0,0)$  is locally asymptotical stability.

**Case III :** For  $E_2(S_{x2}, I_{x2}, 0, 0)$  the Jacobi matrix is

$$J_{E_2} = \begin{vmatrix} -\frac{r}{K}S_{x2} & (-\frac{r}{K} - \beta_1)S_{x2} & 0 & 0 \\ \beta_1I_{x2} & 0 & -k_2I_{x2} & 0 \\ 0 & 0 & k_1\theta S_{x2} + k_2\theta I_{x2} & 0 \\ 0 & 0 & 0 & -d_2 \end{vmatrix}$$

The corresponding characteristic polynomial is

$$D_2(\lambda) = (\lambda + d_2)(\lambda - k_1\theta S_{x2} - k_2\theta I_{x2} + d_1) \left[ \lambda^2 + \frac{r}{K}S_{x2}\lambda + \left(\frac{r}{K} + \beta_1\right)\beta_1S_{x2}I_{x2} \right]$$

Obviously  $\lambda_1 = -d_2 < 0, \lambda_2 = k_2\theta I_{x2} - d_1$ . Let  $f_2(\lambda) = \lambda^2 + \frac{r}{K}S_{x2}\lambda + \left(\frac{r}{K} + \beta_1\right)\beta_1S_{x2}I_{x2}$

Because  $\frac{r}{K} S_{x_2} > 0, \left(\frac{r}{K} + \beta_1\right) \beta_1 S_{x_2} I_{x_2} > 0$ , so  $\lambda_3, \lambda_4$  have negative real part .

When  $k_1 \theta S_{x_2} + k_2 \theta I_{x_2} < d_1, \lambda_2 < 0$ . So  $E_2(S_{x_2}, I_{x_2}, 0, 0)$  is locally asymptotical stability .

**Case IV:** For  $E_3(S_{x_3}, I_{x_3}, S_{y_3}, 0)$  the Jacobi matrix is

$$J_{E_3} = \begin{vmatrix} -\frac{r}{K} S_{x_3} & (-\frac{r}{K} - \beta_1) S_{x_3} & 0 & 0 \\ \beta_1 I_{x_3} & 0 & -k_2 I_{x_3} & 0 \\ 0 & k_2 \theta S_{y_3} & 0 & -\beta_2 S_{y_3} \\ 0 & 0 & 0 & \beta_2 S_{y_3} - d_2 \end{vmatrix}$$

$$D_3(\lambda) = (\beta_2 S_{x_3} - d_2 - \lambda) \left\{ -\lambda^3 - \frac{r}{K} S_{x_3} \lambda^2 - \left[ K^2 \theta I_{x_3} S_{y_3} + \beta_1 \left( \frac{r}{K} + \beta_1 \right) S_{x_3} I_{x_3} \right] \lambda - \frac{k_2^2 \theta r}{K} S_{x_3} I_{x_3} S_{y_3} \right\}$$

There is clearly that characteristic root:  $\lambda_1 = \beta_2 S_{y_3} - d_2$ .

Let

$$f_3(\lambda) = -\lambda^3 - \frac{r}{K} S_{x_3} \lambda^2 - \left[ K^2 \theta I_{x_3} S_{y_3} + \beta_1 \left( \frac{r}{K} + \beta_1 \right) S_{x_3} I_{x_3} \right] \lambda - \frac{k_2^2 \theta r}{K} S_{x_3} I_{x_3} S_{y_3} = 0$$
 That is

$$\lambda^3 + \frac{r}{K} S_{x_3} \lambda^2 + \left[ K^2 \theta I_{x_3} S_{y_3} + \beta_1 \left( \frac{r}{K} + \beta_1 \right) S_{x_3} I_{x_3} \right] \lambda + \frac{k_2^2 \theta r}{K} S_{x_3} I_{x_3} S_{y_3} = 0$$

Denote by :

$$h_1 = \frac{r}{K} S_{x_3}, h_2 = k^2 \theta I_{x_3} S_{y_3} + \beta_1 \left( \frac{r}{K} + \beta_1 \right) S_{x_3} I_{x_3}, h_3 = \frac{k_2^2 \theta r}{K} S_{x_3} I_{x_3} S_{y_3}$$

$$H_1 = h_1 > 0, H_2 = h_1 h_2 - h_3 = \frac{r \beta_1}{K} \left( \frac{r}{K} + \beta_1 \right) S_{x_3}^2 I_{x_3} > 0, H_3 = h_3 H_2 > 0$$

All roots have negative real parts, by Hurwitz criterion.

Hence, we have the following main theorems:

**Theorem 1:**  $E_0(0, 0, 0, 0)$  is a saddle point. When  $\beta_1 K < d_0, E_1(K, 0, 0, 0)$  is locally asymptotical stability.

When  $k_1 \theta S_{x_2} + k_2 \theta I_{x_2} < d_1, E_2(S_{x_2}, I_{x_2}, 0, 0)$  is locally asymptotical stability.

When  $\beta_2 S_{y_3} < d_2, E_3(S_{x_3}, I_{x_3}, S_{y_3}, 0)$  is locally asymptotical stability.

#### 4. GLOBAL STABILITY

**Theorem 2:** The positive equilibria point  $E_4 = (S_{x4}, I_{x4}, S_{y4}, 0)$  of the system (1) is global asymptotical stability.

**Proof:** Take proper Lyapunov function  $V(S_x, I_x, S_y, I_y): R_+^4 \rightarrow R$

$$V(t) = m_1(S_x - S_{x4} - S_{x4} \ln \frac{S_x}{S_{x4}}) + m_2 \left( I_x - I_{x4} - I_{x4} \ln \frac{I_x}{I_{x4}} \right) \\ + m_3(S_y - S_{y4} - S_{y4} \ln \frac{S_y}{S_{y4}}) + m_4 \left( I_y - I_{y4} - I_{y4} \ln \frac{I_y}{I_{y4}} \right)$$

For  $V(t)$  derivation along the system, we have:

$$\dot{V}(t) = m_1 \frac{S_x - S_{x4}}{S_x} \frac{dS_x}{dt} + m_2 \frac{I_x - I_{x4}}{I_x} \frac{dI_x}{dt} + m_3 \frac{S_y - S_{y4}}{S_y} \frac{dS_y}{dt} + m_4 \frac{I_y - I_{y4}}{I_y} \frac{dI_y}{dt} \\ = m_1(S_x - S_{x4}) \left[ r \left( 1 - \frac{S_x + I_x}{K} \right) - \beta_1 I_x - k_1 S_y - r \left( 1 - \frac{S_{x4} + I_{x4}}{K} \right) + \beta_1 I_{x4} + k_1 S_{y4} \right] \\ + m_2(I_x - I_{x4})(\beta_1 S_x - k_2 S_y - d_0 I_x + \beta_1 S_{x4} + k_2 S_{y4} + d_0) \\ + m_3(S_y - S_{y4})(\beta_1 \theta S_x - d_1 - \beta_2 I_y - k_1 \theta S_{x4} + d_1 + \beta_2 I_{y4} - k_2 \theta I_{x4}) \\ + m_4(I_y - I_{y4})(\beta_2 S_y - d_2 - \beta_2 S_{y4} + d_2) \\ = m_1(S_x - S_{x4}) \left[ -\frac{r}{K}(S_x - S_{x4}) - \left( \frac{r}{K} + \beta_1 \right) (I_x - I_{x4}) - k_1(S_y - S_{y4}) \right] \\ + m_2(I_x - I_{x4}) \left[ \beta_1(S_x - S_{x4}) - k_2(S_y - S_{y4}) \right] \\ + m_3(S_y - S_{y4}) \left[ k_1 \theta (S_x - S_{x4}) - \beta_2(I_y - I_{y4}) + k \theta (I_x - I_{x4}) \right] + m_4 \beta_2 (I_y - I_{y4})(S_y - S_{y4}) \\ = -\frac{r}{K} m_1 (S_x - S_{x4})^2 + \left[ \beta_1 m_2 - m_1 \left( \frac{r}{K} + \beta_1 \right) \right] (S_x - S_{x4})(I_x - I_{x4}) \\ + (m_3 k_1 \theta - m_2 k_1)(S_x - S_{x4})(S_y - S_{y4}) + (k_2 \theta m_3 - m_2 k_2)(I_x - I_{x4})(S_y - S_{y4}) \\ + (m_4 \beta_2 - m_3 \beta_2)(S_y - S_{y4})(I_y - I_{y4})$$

Let

$$m_2\beta_1 - m_1\left(\frac{r}{K} + \beta_1\right) = 0, m_3k_1\theta - m_1k_1 = 0, k_2\theta m_3 - m_2k_2 = 0, m_4\beta_2 - m_3\beta_2 = 0.$$

Namely, when take  $m_2 = \frac{1}{\beta}\left(\frac{r}{K} + \beta_1\right)m_1, m_3 = m_4\frac{1}{\theta}m_1, m_1 > 0$ , then

$$\dot{V}(t) = -\frac{r}{K}m_1(S_x - S_{x4})^2 \leq 0.$$

So by the LaSalle in variant set theorems we know that  $E_4(S_{x4}, I_{x4}, S_{y4}, I_{y4})$  are global asymptotical stability.

This paper mainly discusses the prey-predator model with disease in the preys and predators, we get the conditions of local asymptotic and the existence of the boundary balance. We prove the positive balance point

$E_4(S_{x4}, I_{x4}, S_{y4}, I_{y4})$  is global asymptotical stability by constructing Liapunov function.

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