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TRAVELING WAVE SOLUTIONS FOR THE NONLINEAR FRACTIONAL SHARMA-TASSO-OLEVER EQUATION

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ABSTRACT

In this paper, the generalized exp(-Φ(ξ))-expansion method along with the Jumarie's modified Riemann-Liouville derivatives is proposed to solve the nonlinear fractional Sharma-Tasso-Olever (STO) equation. Consequently, the exact solutions are obtained in terms of the trigonometric, exponential, hyperbolic, and rational functions, which confirm the proposed technique is very effectual and easily applicable.

Contribution/Originality: This study contributes in the existing literature on the use of the exp(-Φ(ξ))-expansion method. The method is applied to find the exact solutions to the fractional STO equation for the first time. Consequently, we get some new forms of exact solutions.

1. INTRODUCTION

With the development of nonlinear Science, increasing scholars regard the world around us as a nonlinear system and thus a plenty of nonlinear PDEs are widely used as models in various fields of natural sciences [1, 2]. A particular category of nonlinear PDEs are nonlinear fractional PDEs that have continually appeared in physics, chemistry, biology, polymeric materials, electromagnetic, acoustics, neutron point kinetic model, vibration and control, signal and image processing, fluid dynamics and so on [3-6]. Due to its practicability and complexity, it is important to seek the solutions of nonlinear fractional PDEs and researchers [7-24] have put considerable effort into it. For the purpose of solving problems in the practical application fields, more exact traveling wave solutions to the fractional PDEs seem to be useful. Up to now, a large number of methods have been applied to seek the solutions to nonlinear fractional PDEs, such as the fractional first integral method [11, 12] the fractional sub-equation method [13, 14] the (G'/G)-expansion method [15, 16] the improved (G'/G)-expansion method [17] the functional variable method [18] the fractional modified trial equation method [19] the extended spectral method [20] the variational iteration method [21-24] and so on. It is worth mentioning that Li and He [25, 26] have proposed a fractional complex transformation to convert fractional differential equations into ordinary differential equations (ODEs), which makes the problem simple. It means that the analytical methods devoted to advance calculus can also be applied to the fractional differential equations easily [27].

In recent years, the exp(-Φ(ξ))-expansion method have been implemented by many authors [28-30] to search the exact solutions of the nonlinear PDEs appeared in various fields as mentioned earlier. In these articles [28-30] the exp(-Φ(ξ))-expansion method along with the nonlinear ordinary differential equation (ODE) $\Phi'(\xi)=\exp(-\Phi(\xi))+\mu\exp(\Phi(\xi))+\lambda, \lambda, \mu \in \Re$ have provided some comprehensive solutions to the nonlinear PDEs. In 2015, Hafez and Akbar [31] have applied the exp(-Φ(ξ))-expansion method to solve strain wave equation appeared in microstructured solids by considering $\Phi'(\xi)=\exp(-\Phi(\xi))+\mu\exp(\Phi(\xi))+\lambda,$ $\Phi'(\xi)=-\sqrt{\lambda+\mu\exp(-\Phi(\xi))^2}$ and $\Phi'(\xi)=-\lambda\exp(-\Phi(\xi))-\mu\exp(\Phi(\xi))$ as auxiliary ODEs. Inspired by the above, the nonlinear ODE $\Phi'(\xi)=p\exp(-\Phi(\xi))+q\exp(\Phi(\xi))+r$ have been applied as auxiliary equation for searching more comprehensive solutions to nonlinear PDEs, so-called the generalized exp(-Φ(ξ))-expansion method [27]. Thus, the objective of this paper is to present the generalized exp(-Φ(ξ))-expansion method and implement it to find the exact traveling wave solutions of the fractional STO equation [11, 32, 33]. The proposed exp(-Φ(ξ))-expansion method along with the auxiliary nonlinear ODE provides much more comprehensive solutions and easily applicable to solve the nonlinear PDEs. Moreover, we have tried to generalize this method for finding more comprehensive exact traveling wave solutions to the nonlinear fractional PDEs in this paper. Sometimes this method can give solutions in disguised versions of known solutions that already be obtained by other methods. The superiority of this method over the existing methods is that it provides some new exact traveling wave solutions together with additional free parameters [27]. The algebraic computation of this method in this article is realized with the help of algebraic software, i.e., Mathematica.

The rest of the paper is prepared as follows: In section 2, the definitions of the fractional derivative is introduced concisely and the proposed generalized exp(-Φ(ξ))-expansion method is presented in details. Section 3 presents the application of this method to construct the exact traveling wave solutions of the nonlinear fractional STO equation. In comparison with other methods, the advantage of the proposed method is given in section 4. Conclusions have been drawn in Section 5.

2. DESCRIPTION OF THE METHOD

This section consists of two parts: the basic idea of the fractional derivative and the detailed steps for using the proposed generalized $\exp(-\Phi(\xi))$-expansion method.

Firstly we introduce the definitions of the Jumarie [34] derivatives by the following expression [35]

$$D_x^{\alpha} f(x)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^z (x-\xi)^{-\alpha}(f(\xi)-f(0))d\xi;\ 0<\alpha<1\\ (f^{(n)}(x))^{(\alpha-n)};\ n\le\alpha<n+1, n\ge 1,\end{cases} \qquad (1)$$

where $f(x)$ is a continuous (but not necessarily differential) function of x.

Some important properties and formulas can be deduced from the above, such as [35-37]

$$D_x^{\alpha} z^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \gamma > 0, \tag{2}$$

which will be applied in the following content.

Then we outline the main steps of the generalized $\exp(-\Phi(\xi))$-expansion method for solving nonlinear fractional PDEs. Let us consider the nonlinear fractional PDEs in the following form [27, 35]:

$$f(u, u_x, u_t, D_t^{\alpha} u, D_x^{\alpha} u, D_x^{\beta} u, D_t^{\alpha} D_t^{\beta} u, D_x^{\alpha} D_x^{\beta} u, \ldots\ldots) = 0, \alpha > 0, \beta < 1, \tag{3}$$

where, $u = u(x,t)$ is an unknown function, f is a function of $u(x,t)$, its derivatives and partial fractional derivatives, in which higher order derivatives and nonlinear terms are involved [27, 35]. For the purpose of searching the exact solutions of (3) by an explicit way, we should perform the following four steps:

Step 1. One can use the compound variable ξ to combine the real variables x and y [27, 35]

$$u(x,t) = U(\xi), \ \xi = \frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{4}$$

where k and c are constants.

With the help of Eq. (2) and (4), Eq. (3) can be converted into a nonlinear ODE for $U = U(\xi)$:

$$F(U, U', U'', U''', \cdots\cdots\cdots) = 0, \tag{5}$$

where, F is a function of U and its derivatives, the superscript point out the ordinary derivatives [27] with respect to ξ.

Step 2. We can suppose the traveling wave solutions of Eq. (5) as Hafez and Lu [27]

$$U(\xi) = \sum_{i=0}^{N} A_i \left(e^{-\Phi(\xi)}\right)^i, \ A_N \neq 0 \tag{6}$$

where the coefficients $A_i (0 \leq i \leq N)$ are constants to be evaluated, such that $A_N \neq 0$ and $\Phi = \Phi(\xi)$ satisfies the following first order nonlinear ODE:

$$\Phi'(\xi) = p e^{-\Phi(\xi)} + q e^{\Phi(\xi)} + r, \tag{7}$$

Step 3. If we balance the higher order derivative with the nonlinear terms of the highest order that appeared in Eq. (5), the value of the positive integer N is consequently obtained [27, 35]. For example, suppose the degree of $U = U(\xi)$ is $D[U(\xi)] = n$, then the degree of the other expressions can be determined by the following formula:

$$D[\frac{d^N U(\xi)}{d\xi^N}] = N + n, \quad D[u^N \left(\frac{d^K U(\xi)}{d\xi^K}\right)^S] = nN + S(n+k). \tag{8}$$

Step 4. Substituting Eq. (6) into Eq. (5) and using Eq. (7) rapidly, we can get a function of $\exp(-\Phi(\xi))$ .Then find out the coefficients of same power of $\exp(-\Phi(\xi))$ and define them equal to zero, one can acquire a system of algebraic equations for $A_i\,(0 \le i \le N), k, p, q, r, c$. With the aid of symbolic computation, such as Mathematical, one can account the obtaining system and find out the values of $A_i\,(0 \le i \le N), k, p, q, r, c$.

To give the ultimate solutions of (3), the general solutions of equation (7) have been provided as follows:
Type 1: when $p = 1$, one obtains

$$\Phi(\xi) = \ln\left(-\frac{\sqrt{r^2 - 4q}\tanh\left\{0.5\sqrt{r^2 - 4q}(\xi + \xi_0)\right\}}{2q} - \frac{r}{2q} \right), q \ne 0, r^2 - 4q > 0, \tag{9a}$$

$$\Phi(\xi) = \ln\left(\frac{\sqrt{4q - r^2}\tan\left\{0.5\sqrt{4q - r^2}(\xi + \xi_0)\right\}}{2q} - \frac{r}{2q} \right), q \ne 0, r^2 - 4q < 0, \tag{9b}$$

$$\Phi(\xi) = -\ln\left(\frac{r}{\exp(r(\xi + \xi_0)) - 1} \right), q = 0, r \ne 0, r^2 - 4q > 0, \tag{9c}$$

$$\Phi(\xi) = \ln\left(-\frac{2(r(\xi + \xi_0) + 2)}{r^2(\xi + \xi_0)} \right), q \ne 0, r \ne 0, r^2 - 4q = 0, \tag{9d}$$

Type 2: when $r = 0$, one obtains

$$\Phi(\xi) = \ln\left(\sqrt{\frac{p}{q}}\tan\left[\sqrt{pq}\,(\xi + \xi_0)\right] \right), p > 0, q > 0, \tag{9e}$$

$$\Phi(\xi) = \ln\left(-\sqrt{\frac{p}{q}}\tan\left[\sqrt{pq}\,(\xi - \xi_0)\right] \right), p < 0, q < 0, \tag{9f}$$

$$\Phi(\xi) = \ln\left(\sqrt{\frac{-p}{q}}\tanh\left[\sqrt{-pq}\,(\xi - \xi_0)\right] \right), p > 0, q < 0, \tag{9g}$$

$$\Phi(\xi) = \ln\left(-\sqrt{\frac{-p}{q}}\tanh\left[\sqrt{-pq}\,(\xi + \xi_0)\right] \right), p < 0, q > 0. \tag{9h}$$

Type 3: when $q = 0$ and $r = 0$, one obtains

$$\Phi(\xi) = \ln\{p(\xi + \xi_0)\}, \tag{9i}$$

where ξ_0 is the integrating constant.

Follow the above steps, we are obtained the multiple explicit solutions of nonlinear fractional PDE (3) by combining the equations (4), (6) and (9).

3. APPLICATION TO NONLINEAR FRACTIONAL STO EQUATION

To illustrate the feasibility of the proposed method, we employ the steps in section 2 to solve a nonlinear fractional PDE.

Let us consider the space-time fractional STO equation as follows [11, 32, 33]

$$D_t^{\alpha} u + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0, t > 0, 0 < \alpha \le 1. \tag{10}$$

In Song, et al. [32] the author obtained a rational approximate solution of (10) by the use of the variational iteration method, the Adomain decomposition method, and the homotopy perturbation method. While in Lu [11] the first integral method was used for obtaining the exact solutions of (10). As a result, some trigonometric function and hyperbolic function solutions are achieved. In Zheng [33] more exact solutions of (10) are founded by the Exp-function method, which is different from the generalized $\exp(-\Phi(\xi))$ -expansion method proposed in this article.

To begin with, we take the traveling wave transformation in Eq. (10),

$$u(x,t)=U(\xi),\ \ \xi=k\left(x+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right), \tag{11}$$

where k and c are constants. Then Eq. (10) is reduced into a nonlinear ODE as follows:

$$kcU'+3ak^2(U')^2+3akU^2U'+3ak^2UU''+ak^3U'''=0, \tag{12}$$

where primes denote the differentiation with respect to ξ. Integrating the eq. (12) once and setting the integration constant to zero for simplicity, we get

$$cU+3akUU'+aU^3+ak^2U''=0. \tag{13}$$

According to the balancing principle, we have $N=1$.Then the solution of (13) can be written as

$$U(\xi)=A_0+A_1e^{-\Phi(\xi)}, \tag{14}$$

where A_0, A_1 are constants to be determined later and $\Phi(\xi)$ satisfies the auxiliary nonlinear ODE (7).

Next, we need to carry out Step 4 in previous section, i.e., substituting Eq. (14) into eq. (13) and using (7) frequently, the left-hand side of Eq. (13) becomes a polynomial in $e^{-\Phi(\xi)}$, and the right-hand side of Eq. (13) is zero. Thus, setting the coefficients of this $(\exp(-\Phi(\xi)))^i, (i=0,1,2,3)$ to zero, we get a system of algebraic equations as follows:

$$\begin{cases}-3akpA_1^2+aA_1^3+2ak^2p^2A_1=0,\\ -3ak(pA_0A_1+rA_1^2)+3aA_0A_1^2+3arpk^2A_1=0,\\ cA_1-3ak(rA_0A_1+qA_1^2)+3aA_0^2A_1+2apqk^2A_1+ak^2r^2A_1=0,\\ cA_0-3akqA_0A_1+aA_0^3+ak^2qrA_1=0.\end{cases} \tag{15}$$

As we can see in (15), the number of unknown variables is a little bit more in the algebraic equations. It is too complicated if we solve (15) by calculations, thus, we can solve it with the help of mathematical software, such as Mathematica. Then we can have the following sets of solutions:

Set 1: $A_0=\frac{kr}{2}, A_1=kp, k=k, c=\frac{1}{4}ak^2(4pq-r^2)$ (16)

Set 2: $A_0=\frac{k}{2}\left(r\pm\sqrt{r^2-4pq}\right), A_1=kp, k=k, c=ak^2(4pq-r^2)$ (17)

where k, p, q and r are arbitrary constants.

According to **Set 1**, and considering Eq. (9), (11), (14) at the same time, we find out the following explicit solutions to the fractional STO equation (10):

For Type 1:

$$u_1(x,t)=\frac{kr}{2}-k\frac{2q}{\sqrt{r^2-4q}\tanh\left[0.5\sqrt{r^2-4q}\left(\xi+\xi_0\right)\right]+r},\quad q\neq 0, r^2-4q>0, \tag{18}$$

$$u_2(x,t)=\frac{kr}{2}+k\frac{2q}{\sqrt{4q-r^2}\tan\left[0.5\sqrt{4q-r^2}\left(\xi+\xi_0\right)\right]-r},\quad q\neq 0, r^2-4q<0, \tag{19}$$

$$u_3(x,t)=\frac{kr}{2}+k\frac{r}{\exp\left(r\left(\xi+\xi_0\right)\right)-1}, q=0,\ r^2-4q>0, \tag{20}$$

$$u_4(x,t)=\frac{kr}{2}-k\frac{r^2\left(\xi+\xi_0\right)}{2r\left(\xi+\xi_0\right)+4}, q\neq 0, r\neq 0,\ r^2-4q=0, \tag{21}$$

where $\xi=k\left(x+\frac{ak^2\left(4q-r^2\right)}{4\Gamma(1+\alpha)}t^{\alpha}\right)$.

For Type 2:

$$u_5(x,t)=k\sqrt{pq}\cot\left[\sqrt{pq}\left(\xi+\xi_0\right)\right], p>0, q>0, \tag{22}$$

$$u_6(x,t)=k\sqrt{pq}\cot\left[\sqrt{pq}\left(\xi-\xi_0\right)\right], p<0, q<0,, \tag{23}$$

$$u_7(x,t)=k\sqrt{-pq}\coth\left[\sqrt{-pq}\left(\xi-\xi_0\right)\right], p>0, q<0, \tag{24}$$

$$u_8(x,t)=k\sqrt{-pq}\coth\left(\sqrt{-pq}\left(\xi+\xi_0\right)\right), p<0, q>0, \tag{25}$$

Where $\xi=k\left(x+\frac{ak^2pq}{\Gamma(1+\alpha)}t^{\alpha}\right)$.

For Type 3:

$$u_9(x,t)=\frac{k}{kx+\xi_0}, q=0,\ r=0. \tag{26}$$

The figures of solutions $u_1(x,t)$ and $u_3(x,t)$ are given as follows:

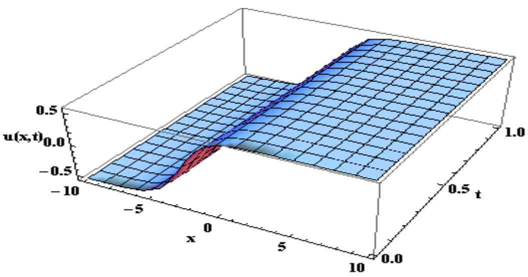

Figure-1. Exact traveling wave solutions of u_1 at $a=1, k=0.5, q=1, r=3, \xi_0=0.5, \alpha=1/3,$ in the interval [-10,10] and time in the interval [0,1].
Source: The figure is plotted by Mathematica.

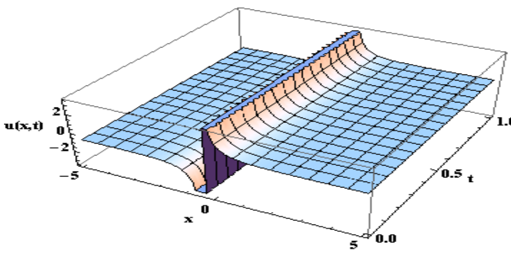

Figure-2. Exact traveling wave solutions of u_3 at $a=1, k=1, q=0.5, r=1.5, \xi_0=0.5, \alpha=0.2,$ in the interval [-5,5] and time in the interval [0,1].
Source: The figure is plotted by Mathematica.

Similarly, due to **Set 2**, the solutions of Eq.(10) are as follows:

For Type 1:

$$u_{10}(x,t)=k\left\{\frac{r\pm\sqrt{r^2-4q}}{2}-\frac{2q}{\sqrt{r^2-4q}\tanh\left[0.5\sqrt{r^2-4q}(\xi+\xi_0)\right]+r}\right\},q\neq 0,r^2-4q>0, \quad (27)$$

$$u_{11}(x,t)=k\left\{\frac{r\pm\sqrt{4q-r^2}}{2}+\frac{2q}{\sqrt{4q-r^2}\tan\left[0.5\sqrt{4q-r^2}(\xi+\xi_0)\right]-r}\right\},q\neq 0,r^2-4q<0, \quad (28)$$

$$u_{12}(x,t)=k\left\{r+\frac{r}{\exp(r(\xi+\xi_0))-1}\right\},q=0,r\neq 0,r^2-4q>0, \quad (29)$$

$$u_{13}(x,t)=k\left(\frac{r}{2}-\frac{r^2(\xi+\xi_0)}{2r(\xi+\xi_0)+4}\right),q\neq 0,r\neq 0,r^2-4q=0, \quad (30)$$

where $\xi=k\left(x+\frac{ak^2(4q-r^2)}{\Gamma(1+\alpha)}t^{\alpha}\right)$.

For Type 2:

$$u_{14}(x,t)=k\sqrt{pq}\left\{\pm 1+\cot\left[\sqrt{pq}(\xi+\xi_0)\right]\right\}, p>0,q>0, \quad (31)$$

$$u_{15}(x,t)=k\sqrt{pq}\left\{\pm 1+\cot\left[\sqrt{pq}(\xi-\xi_0)\right]\right\}, p<0,q<0, \quad (32)$$

$$u_{16}(x,t)=k\sqrt{-pq}\left\{\pm 1+\coth\left[\sqrt{-pq}(\xi-\xi_0)\right]\right\}, p>0,q<0, \quad (33)$$

$$u_{17}(x,t)=k\sqrt{-pq}\left\{\pm 1+\coth\left[\sqrt{-pq}(\xi+\xi_0)\right]\right\}, p<0,q>0, \quad (34)$$

where $\xi=k\left(x+\frac{4ak^2pq}{\Gamma(1+\alpha)}t^{\alpha}\right)$.

For Type 3:

$$u_{18}(x,t)=\frac{k}{kx+\xi_0},q=0,r=0. \quad (35)$$

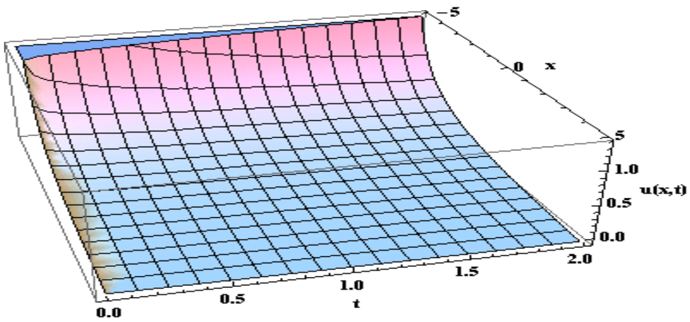

Figure-3. Exact traveling wave solutions of u_{13} at $a=1,k=1,r=2,\xi_0=0.5,\alpha=0.5,$ in the interval [-5,5] and time in the interval [0,2].
Source: The figure is plotted by Mathematica.

4. DISCUSSION

As we can see in the given example, through implementing the straightforward steps, a considerable number of solutions are achieved in various forms, which can be regarded as the superiority of the generalized $\exp(-\Phi(\xi))$ -expansion method over other methods.

In [28-30] the $\exp(-\Phi(\xi))$ -expansion method is used to find the exact solutions of PDEs, the auxiliary differential equation selected in these papers is $\Phi'(\xi)=\exp(-\Phi(\xi))+\mu\exp(\Phi(\xi))+\lambda, \lambda, \mu \in \Re$, i.e., $p=1$. Under the circumstances, some solutions may be ignored. While the auxiliary ODE in this article is $\Phi'(\xi)=p\exp(-\Phi(\xi))+q\exp(\Phi(\xi))+r$. When $p=1$, the exp(-Φ(ξ))-expansion method is only a special case of the method proposed in this article. Due to the introduction of the free parameter p , the solutions become more genera.

In Lu [11] the first integral method is applied to seek the exact solutions of fractional STO equation, as a result, the solutions are appeared in the form of hyperbolic functions and trigonometric functions. Considering the solutions (22)~(25), (31)~(34) in this article, if the parameters are set to particular values, the results are in accordance with the solutions (53)~(56) in Lu [11] . In addition, solutions (18), (19), (20), (21), (26), (27), (28), (29), (30), (35) are new exact traveling wave solutions to the nonlinear fractional STO equation.

In Zheng [33] the fractional STO equation is solved by the Exp-function method, the solutions in Zheng [33] are obtained in terms of exponential, hyperbolic and trigonometric functions. While the solutions denoted by (18)~(20), (22)~(25), (27)~(29), (31)~(34) in this paper are obviously different from the results in Zheng [33]. In addition, the solutions denoted by (21), (26), (30), (35) in this paper are in the form of rational functions, which can be seen as new solutions obtained by the generalized exp(-Φ(ξ))-expansion method.

Through the above comparative analysis, it is a remarkable fact that the proposed method is a practical mathematical method to search for the exact solutions of nonlinear fractional PDEs.

5. CONCLUSION

In this paper, the generalized exp(-Φ(ξ))-expansion method along with the Jumarie's modified Riemann-Liouville derivatives has been successfully applied to the nonlinear fractional STO equation for the first time. The Discussion Section shows that the proposed method can give more solutions in general forms compared with other methods. Besides, this method is straightforward and easily applicable. Based on this advantage, we can also apply it to many other nonlinear fractional PDEs. The exact solutions obtained via this method have its great potential in the further analysis, such as stability analysis and compare with numerical solutions arises in various fields of applied mathematics and mathematical physics [27].

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