International Journal of Mathematical Research
2017 Vol. 6, No. 2, pp. 46-52
ISSN(e): 2306-2223
ISSN(p): 2311-7427
DOI: 10.18488/journal.24.2017.62.46.52

PERIOD MONOTONICITY FOR WEIGHT-HOMOGENEOUS DIFFERENTIAL SYSTEMS

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Article History
Received: 20 March 2017
Revised: 30 May 2017
Accepted: 23 June 2017
Published: 13 July 2017

Keywords
Integrability
Centre
Period function
Quasi-homogeneous polynomial
Center-focus problem
Vector field.

ABSTRACT

In this article, integrability, center, and monotonicity of associated period function for -quasi-homogeneous vector fields are investigated. We are concerned with family of vector field given by sum, finite or infinite number of quasi-homogeneous polynomials not necessarily to be sharing the same weights. The investigation is done by utilizing method of computing focal values. As an application of the result, a particular family of (p, q)-quasi-homogeneous vector field is studied to find conditions for center, monotonicity and consequently an explicit form for the associated period function.

1. INTRODUCTION

The quasi-homogeneous (and in general nonhomogeneous) polynomial function is defined as follows,

Definition 1. *Let* p, q, n *be positive integers. The polynomial* $P_{n,(p,q)}(x, y)$ *is called a* (p, q) *-quasi-homogeneous polynomial of weight degree* n *if*

$$P_{n,(p,q)}(\lambda^p x, \lambda^q y) = \lambda^n P_{n,(p,q)}(x, y)$$

for all real number λ.

Notice that $P_{n,(p,q)}(x, y)$ can be written as,

$$P_{n,(p,q)}(x, y) = \sum_{ip+jq=n} p_{ij} x^i y^j$$

Or as

$$P_{n,(p,q)}(x, y) = \sum_{k=1}^{r} p_{i_k j_k} x^{i_k} y^{j_k}, \quad i_k p + j_k q = n$$

$$= (x^{i_1}, \ldots, x^{i_r}) \begin{pmatrix} p_{i_1,j_1} & & & & \\ & . & & 0 & \\ & & . & & \\ & 0 & & . & \\ & & & & p_{i_r,j_r} \end{pmatrix} \begin{pmatrix} y^{j_1} \\ . \\ . \\ . \\ y^{j_r} \end{pmatrix}$$

A planar polynomial vector field of the form

$$X(x,y) = (\ P_{n,(p,q)}(x,y), Q_{n,(p,q)}(x,y)\ )$$

where $P_{n,(p,q)}(x,y), Q_{n,(p,q)}(x,y)$ are (p,q)-quasi-homogeneous polynomials of weight degree n, is called (p,q)-quasi-homogeneous of quasi-degree n vector field. Notice that homogeneous vector fields of degree n are quasi-homogeneous of quasi-degree n and weight (1, 1).

The quasi-homogeneous polynomial differential systems have been studied from many different point of view, one of these studies is the Centre, see for instance [1]; [2]; [3]; [4]. But up to now there was not an algorithm for constructing all the quasi-homogeneous polynomial differential systems for a given degree. In this paper we are concerned with (p,q)-quasi-homogeneous vector field having a degenerate critical point, at the origin given by sum of quasi-homogeneous polynomials. We study the integrability, center conditions, and the monotonicity of associated period function and moreover give a closed form for the period function of (p,q)-quasi-homogeneous vector fields of particular case.

A critical point is called a Centre if it has a punctured neighborhood full of periodic orbits. The largest of such neighborhood is called the period annulus of the Centre. If the eigenvalues of the linear part of X at the Centre are not purely imaginary, then the Centre is called degenerate. This is our case since n >1. In the period annulus of a center the so-called period function T(x) gives the least period of the periodic solution passing through the point with coordinates (x,0)=(r,0) inside the period annulus of the Centre. If all periodic solutions inside the period annulus of the Centre have the same period it is said that the Centre is isochronous. For more details on characterization of isochronocity see [5] and references therein. For a Centre that is not isochronous, any value r>0 for which $\frac{dT(r)}{dr} = 0$ is called a critical period. When a critical point is degenerate, its Centre problem has also been investigated by many authors, see for example [6]; [7] and references therein. Recall that a critical point is called monodromic if there are no orbits tending or leaving the point with a certain direction. For analytic vector fields, monodromic points are either Centre or focus, and the problem of distinguishing between both options is called the Centre -focus problem. In order to have a Centre at the origin we only need to guarantee that the origin is monodormic and moreover, that some definite integral, that can be obtained from the expression in quasi-homogeneous polar coordinates, is zero.

We write the vector field associated with the differential system (1.1) as X(x, y). We give conditions on the parameters of the system in order to get integrable system and to study its period function on the period annulus of the origin when we assume that the differential equation associated to X has a degenerate Centre at this point.

In this article, we are interested in characterizing integrability and monotonicity of period function of degenerate Centre for certain classes of planar polynomial differential system given by sum, finite or infinite number, of quasi-homogeneous polynomials of the form

$$\dot{x} = P_{n,(p,q)}(x,y) + px\sum_{i=1}^{N} r^{\alpha_i} R_{l_i,(p,q)}(x,y) \quad .$$

$$\dot{y} = Q_{m,(p,q)}(x,y) + qy\sum_{i=1}^{N} r^{\alpha_i} R_{l_i,(p,q)}(x,y) \qquad (1.1)$$

and the quasi-homogeneous polynomials $P_{n,(p,q)}(x,y), Q_{m,(p,q)}(x,y)$, and $R_{l_{ii},(p,q)}(x,y)$ are not necessarily to be sharing same weight degree. In the literature the authors investigated classes of quasi-homogeneous polynomial systems with a given weight degree sharing all the parts of the system, see for instance [8] and references therein. Our result extends the homogeneous case as a particular case.

2. MAIN RESULTS

We consider a class of differential system given in 1.1, to study the integrability by giving an explicit from for its first integral and then investigate conditions for Centre for some subclass and monotonicity of the associated period function. For sufficiently small h>0, the solution of system 1.1 which satisfy initial condition $x|_{t=0} = h^p, \quad y|_{t=0} = 0$ goes around the origin at the neighborhood of $x^{\frac{2}{p}} + y^{\frac{2}{q}} = r^2$, then phase curves in the neighborhood of origin of system 1.1 can be studied by the following polar transformation

$$\begin{aligned} x &= r^p \cos^p \theta \\ y &= r^q \sin^q \theta \end{aligned} \qquad (2.1)$$

where p, q are positive odd integers ≥1. Choosing odd is to allow x and y take negative values. Imposing these parameters p, q gives one possibility of maneuver in choosing desired form for desired purpose. The following Theorem is the first result.

Theorem 1. *The function*

$$H(x,y)=(x^{\frac{2}{p}} + y^{\frac{2}{q}})^{\frac{n-p-a}{2}} Exp[(p+a-n)\int^{\arctan\frac{y^{\frac{1}{q}}}{x^{\frac{1}{p}}}} \frac{f(u)}{g(u)}du] - \qquad (2.2)$$

$$(p+a-n)\int^{\arctan\frac{y^{\frac{1}{q}}}{x^{\frac{1}{p}}}} [\frac{f(u)}{g(u)} Exp(p+a-n)\int^{u} \frac{f(\omega)}{g(\omega)}d\omega]du$$

where $a = \alpha_i + l_i, \ i = 1,\dots,N,$ *is first integral for system1.1.*

Proof. The system 1.1, by making use of the generalized polar coordinates 2.1, is transformed to

$$\dot{r} = [\frac{1}{p}\cos^{2-p}\theta P_{n,(p,q)}(\cos^p\theta,\sin^q\theta) +$$

$$\frac{1}{q}\sin^{2-q}\theta Q_{m,(p,q)}(\cos^p\theta,\sin^q\theta)]r^{n-p+1} + \qquad (2.3)$$

$$[\sum_{i=1}^{N} R_{l_i,(p,q)}(\cos^p\theta,\sin^q\theta)]r^{1+a}$$

$$\dot{\theta} = [\frac{-1}{p}\cos^{1-q}\theta\sin\theta P_{n,(p,q)}(\cos^p\theta,\sin^q\theta) +$$

$$\frac{1}{q}\cos^{1-q}\theta\sin^{2-q} Q_{m,(p,q)}(\cos^p\theta,\sin^q\theta)]r^{n-p}$$

provided that $n-p = m-q$, $\alpha_i + l_i = a$ for $i = 1,2,\dots,N.$

Then for θ where $g(\theta) \neq 0$ we get

$$\frac{dr}{d\theta}=\frac{f(\theta)}{g(\theta)}r+\frac{h(\theta)}{g(\theta)}r^{1+a-n+p} \quad (2.4)$$

where

$$f(\theta)=\frac{1}{p}\cos^{2-p}\theta P_{n,(p,q)}(\cos^p\theta,\sin^q\theta)+$$

$$\frac{1}{q}\sin^{2-q}\theta Q_{m,(p,q)}(\cos^p\theta,\sin^q\theta)$$

$$g(\theta)=\frac{-1}{p}\cos^{1-q}\theta\sin\theta P_{n,(p,q)}(\cos^p\theta,\sin^q\theta)+ \quad (2.5)$$

$$\frac{1}{q}\cos\theta\sin^{1-q}Q_{m,(p,q)}(\cos^p\theta,\sin^q\theta)$$

$$h(\theta)=\sum_{i=1}^{N}R_{l_i,(p,q)}(\cos^p\theta,\sin^q\theta)$$

Equation 2.4 is Bernoulli differential equation and it is transformed to a linear one by making use of classical transformation

$$\rho=r^{n-p-a} \quad (2.6)$$

whose solution given by

$$\rho Exp[(p+a-n)\int^{\theta}\frac{f(u)}{g(u)}du]=\ (p+a-n)\int^{\theta}[\frac{f(u)}{g(u)}Exp(p+a-n)\int^{u}\frac{f(\omega)}{g(\omega)}d\omega]du \quad (2.7)$$

Therefore the expression given in (2.2) is a first integral for the system 1.1. This completes the proof.

Theorem 2. *For system 1.1, if*

$$Exp[\int^{\theta}\frac{f(u)}{g(u)}du]\neq 0$$

for $\theta\in[0,2\pi]$, then the critical point is monodromic.

Proof. From the proof of the Theorem 1, because $g(\theta)\neq 0$ for $\theta\in[0,2\pi]$, the solution of system 2.4 satisfying the initial condition

$$x\big|_{t=0}=h^p \quad (2.8)$$

For sufficiently small h is a power series of h with non-zero radius of convergence when $|\theta|<4\pi$.

Let

$$r=\tilde{r}(\theta;h)=\sum_{k=1}^{\infty}\upsilon_k(\theta)h^k \quad (2.9)$$

Provided

$$\upsilon_1(0)=1,\qquad \upsilon_k(0)=0,\qquad k=2,3,...$$

Substitute 2.9 into 2.4 to get

$$\upsilon_1'(\theta)h+\upsilon_2'(\theta)h^2+...=\frac{f(\theta)}{g(\theta)}[\upsilon_1(\theta)h+\upsilon_2(\theta)h^2+...]+\frac{h(\theta)}{g(\theta)}[\upsilon_1(\theta)h+\upsilon_2(\theta)h^2+...]^{1+a-n+p}$$

For a-n+p>0, equating the corresponding coefficients gives

$$\int\frac{d\upsilon_1}{\upsilon_1}=\int\frac{f(\theta)}{g(\theta)}d\theta$$

Hence

$$\upsilon_1(\theta)=Exp[\int^{\theta}\frac{f(u)}{g(u)}du]$$

So the critical point of system 1.1 is monodromic if $\upsilon_1(\theta)$ does not vanish in [0,2π]. Means the origin of system 1.1 is either focus or Centre. This completes the proof.

Moreover, the Poincare' successive function in the neighborhood of the origin (regarding that $\upsilon_1(0)=\upsilon_1(2\pi)=1$) is given by

$$\Delta(h)=\tilde{r}(2\pi;h)-\tilde{r}(0;h)$$

$$=\sum_{k=1}^{\infty}\upsilon_k(2\pi)h^k-h$$

$$=\sum_{k=2}^{\infty}\upsilon_k(2\pi)h^k$$

Theorem 3. *For the system 1.1 with $n-p-a\neq 0$, if the expression*

$$Exp[(p+a-n)\int^{\theta}\frac{f(u)}{g(u)}du]\int_0^{\theta}[\frac{f(u)}{g(u)}Exp[(p+a-n)\int^{u}\frac{f(\omega)}{g(\omega)}d\omega]du \qquad (2.10)$$

does not vanish, bounded and periodic on [0,2π], then the solution of system 1.1 which satisfies initial condition 2.8 is closed orbit surrounding the origin.

Proof. It is easy to say that the function

$$r(\theta)=[(n-p-a)Exp(n-p-a)\int^{\theta}\frac{f(u)}{g(u)}du]\int_0^{\theta}\frac{f(u)}{g(u)}Exp[(p+a-n)\int^{u}\frac{f(\omega)}{g(\omega)}d\omega]du$$

$$+\left(h\int^{\theta}\frac{f(u)}{g(u)}du\right)^{n-p-a}]^{\frac{1}{n-p-a}} \qquad (2.11)$$

is the unique solution of the equation 2.4 satisfying the initial condition 2.8, where h is positive real number.

From the assumptions on the expression 2.10 we conclude that r (θ) is not vanishing, bounded and periodic on [0,2π]. Hence the solution is closed orbit surrounding the origin.

Applying Theorem 1 with assumption n = a + p we can study the Centre and the associated period function of the following system

$$\dot{x}=P_{a+p,(p,q)}(x,y)+px\sum_{i=1}^{N}r^{\alpha_i}R_{l_i,(p,q)}(x,y) \quad .$$

$$\dot{y} = Q_{a+q,(p,q)}(x,y) + qy\sum_{i=1}^{N} r^{\alpha_i} R_{l_i,(p,q)}(x,y) \qquad (2.12)$$

where $a = \alpha_i + l_i$, for $i = 1,2,\ldots,N.$

Theorem 4. *For system 2.12 ,*
A) If, for the convergent improper integral

$$\int_0^{2\pi} \frac{f(\theta)+h(\theta)}{g(\theta)} d\theta = 0$$

then the system has Centre at the origin.
B) If the system has Centre at the origin under the above condition A), then its associated period function is monotonic decreasing. Moreover it can be written as

$$T(h,0) = T\ h^{-a}$$

for $h \in R^+$*, and some nonzero constant T.*
Proof. Using the generalized polar coordinates 2.1 we can write system 2.12 as

$$\dot{r} = [f(\theta)+h(\theta)]r^{a+1}$$

$$\dot{\theta} = g(\theta)r^{a} \qquad (2.13)$$

where f, g, and h are given in 2.5.
It is clear that g(θ) ≠ 0 for all θ∈[0,2π], then the critical point of the system is monodromic. And then the system 2.13 can be written as

$$\frac{dr}{d\theta} = \frac{f(\theta)+h(\theta)}{g(\theta)} r \qquad (2.14)$$

The solution of this differential equation is given by

$$r(\theta; r_0) = r_0 Exp \int_0^{\theta} \frac{f(u)+h(u)}{g(u)} du \qquad (2.15)$$

where $r_0 = r\ (\ 0\ ,\ r_0\ )$ is the initial condition at θ=0 and $r_0 \succ 0$.
The Centre condition is $r\ (\ 0\ ;\ r_0\ ) = r\ (\ 0\ ;\ r_0\ )$, which implies

$$\int_0^{2\pi} \frac{f(\theta)+h(\theta)}{g(\theta)} d\theta = 0$$

This completes the proof of A).

If we let $\widetilde{T}(r_0)$ denote the period of the orbit passing through the point $(r_0{}^{p},0)$ with the generalized polar coordinates 2.1 regarding the second equation of 2.13 we obtain

$$\widetilde{T}(r_0) = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}}$$

$$= \int_0^{2\pi} \frac{d\theta}{g(\theta) r^a(\theta; r_0)}$$

From equation 2.15 we get

$$\tilde{T}(r_0) = \int_0^{2\pi} \frac{d\theta}{g(\theta)\, Exp[a \int_0^{\theta} \frac{f(u)+h(u)}{g(u)} du}] \frac{1}{r_0^{\,a}}$$

Hence

$$T(h,0) = T\, h^{-a}$$

Where $T = \int_0^{2\pi} \frac{d\theta}{g(\theta)\, Exp[a \int_0^{\theta} \frac{f(u)+h(u)}{g(u)} du}$ is nonzero constant, h ∈ R^+. This completes the proof of B).

Funding: The authors would like to express their gratitude to the University of Sharjah for support.
Competing Interests: The authors declare that they have no competing interests.
Contributors/Acknowledgement: Both authors contributed equally to the conception and design of the study.

REFERENCES

[1] A. Algaba, E. Freire, E. Gamero, and C. Garca, "Monodromy, centre-focus and integrability problems for quasi-homogeneous polynomial systems," *Nonlinear Analysis*, vol. 72, pp. 1726-1736, 2010. *View at Google Scholar* | *View at Publisher*

[2] A. Algaba, N. Fuentes, and C. Garca, "Centre of quasi-homogeneous polynomial planar systems," *Nonlinear Analysis: Real World Applications*, vol. 13, pp. 419-431, 2012. *View at Google Scholar* | *View at Publisher*

[3] M. J. Alvarez, A. Gasull, and R. Prohens, "Global behavior of the period function of the sum of two quasi-homogeneous vecto fields," *Journal of Mathematical Analysis and Applications*, vol. 449, pp. 1553-1569, 2017. *View at Google Scholar* | *View at Publisher*

[4] L. Tao, L. Feng, Y. Liu, and L. Shimin, "Center-focus determination and limit cycles bifurcation for $ p: q $ homogeneous weight singular point," *arXiv preprint arXiv:1608.08890*, 2016.

[5] J. Chavarriga and M. Sabatini, "A survey of isochronous centres," *Qualitative Theory of Dynamical Systems*, vol. 1, pp. 1-70, 1999. *View at Google Scholar*

[6] J. Gine, "On the degenerate centre problem," *International Journal of Bifurcation and Chaos Applied Sciences and Engineering*, vol. 21, pp. 1382-1392, 2011.

[7] J. Gine and J. Llibre, "On the centre conditions for analytic monodromic degenerate singularities," *International Journal of Bifurcation and Chaos Applied Sciences and Engineering*, vol. 22, p. 1250303, 2012. *View at Google Scholar* | *View at Publisher*

[8] Y. Xiong, M. Han, and Y. Wang, "Centre problems and limit cycle bifurcations in a class of quasi-homogeneous system," *International Journal of Bifurcation and Chaos*, vol. 25, p. 1550135, 2015. *View at Google Scholar* | *View at Publisher*