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LOBATTO-RUNGE-KUTTA COLLOCATION AND ADOMIAN DECOMPOSITION METHODS ON STIFF DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we show the parallel of Adomian Decomposition Method (ADM) and Lobatto-Runge-Kutta Collocation Method (LRKCM) on first order initial value stiff differential equations. The former method provided closed form solutions while the latter gave approximate solutions. We illustrated these findings in two numerical examples. ADM solutions were in series form while those of LRKCM gave sizeable absolute error. We further visualized our findings in respective plots to show the great potentials of ADM over LRKCM in providing analytical solutions to stiff differential equations.

Contribution/Originality: This study contributes in showing the originality of ADM in obtaining exact solution to Stiff differential equations, while LRKCM provided approximate solution whose accuracy depended on step size.

1. INTRODUCTION

The exact solution of a stiff differential equation is typically associated with an exponent that has a large magnitude. It include a term that decay exponentially to zero as the independent variable increases, but whose derivative is much greater in magnitude than the term itself. This class of differential equation, in application, arise from phenomena with widely differing time (independent variable) scales. It is very common as mathematical models in physical and biological sciences. They occur in many fields of engineering science particularly in studies of electrical circuits, vibrations, chemical reactions and so on. There are ubiquitous in weather predictions, astrochemical kinetics, control systems and electronics. In general, it application is wide in industrial areas.

A differential equation

$$y' = f(t, y) \tag{1}$$

is stiff if the exact solution include a term that decays exponentially to zero as t increases. Suppose such a term is $e^{-\lambda t}$, where λ is a large positive constant. The kth derivative of this term is $e^{k}e^{-\lambda t}$, and the character e^{k} forces this derivative to decay transiently to zero than $e^{-\lambda t}$ as t grows unbounded. The exact solutions of stiff differential equation (1) are extremely stable but the numerical ones can be extremely unstable with large step size. Numerical methods currently used in literature for obtaining approximate solutions are implicit Euler's and Runge-Kutta method see Haier and Wanner [1]. Explicit Euler's method is also used with unreliable result, finite difference and finite element as well. Others are nth order Taylor's method, linear multistep method like Adams-Bashforth which is implemented as predictor and Adams-Moulton method which is implemented as corrector. The two methods are regarded as predictor-corrector pair. Direct higher order solver like Runge-kutta and LRKCM are also used, see Butcher [2]; Lie and Norsett [3]. All the aforementioned numerical methods are base on obtaining approximate solutions, an approach that is sometimes expensive when higher accuracy is required. The truncation errors for all methods are large and their absolute errors have sizeable values which may not be useful for practical applications. In this paper, we show the great advantages ADM has over LRKCM when used in obtaining solutions to stiff differential equations. The ADM, in this analysis, gave exact solutions to stiff differential equation (1) in series form; while LRKCM gave approximate solutions with errors that increase as the step-size increased. We depict clearly our findings in respective plotted graphs facilitated by Maple software. The use of LRKCM on equation (1) has also been reported by Yakubu, et al. [4].

2. BRIEF CONCEPT OF LRKCM AND ADM

2.1. LRKCM

Consider the general form stiff differential equation as given in equation (1), y = y(t) with the conditions given as

$$y(t_0) = y_0, (t_0 \le t \le b) \tag{2}$$

$y:[t_0, b] \to \Re^n$ and in Yakubu, et al. [4] the continuous multistep collocation approximation formula defined for $[t_0, b]$ is given as

$$y = \sum_{j=0}^{x-1} \alpha_j(t) y_{n+j} + h \sum_{j=0}^{s-1} \beta_j(t) f_{n+j} \tag{3}$$

where

x = number of interpolation points, t_j, j = 0, 1, 2, . . ., x -1

s = distinct collocation points with $\bar{t}_j \in [t_0, b]$, j = 0, 1, . . . s - 1)

h = equally spaced step size (it can also be a variable)

Assuming that the stiff differential equation (1) has only one solution, $\alpha_j(t)$ and $h\beta_j(t)$ are polynomials given as

$$\alpha_j(t) = \sum_{i=0}^{x+s-1} \alpha_{j,i+1} t^i \tag{4}$$

$$h\beta_j(t) = \sum_{i=0}^{x+s-1} h\beta_{j,i+1} t^i \tag{5}$$

where $\alpha_{j,i+1}$ and $\beta_{j,i+1}$ are constant coefficients to be determined. y(t) in (3) is expanded using Taylor series about t, as it is done in linear multistep method, and collecting powers of h to obtain the LRKCM. The resulting multistep collocation and interpolation matrix affects the efficiency, accuracy and stability property of equation (3).

Yakubu, et al. [4] further examined how the constant $\alpha_{j,i+1}$ and $\beta_{j,i+1}$ for his new LRKCM. This was done by deriving multistep collocation method as continuous single finite difference formula of non-uniform order six based on Lobatto points. This was recovered from the popular Lobatto IIIA. Equation (3) becomes a special class of multistep collocation method, for details on LRKCM see Yakubu, et al. [4].

2.2. ADM

Considering an abstract stiff differential equation (1), assuming f is analytic near $y = y_0$, (1) becomes the Volterra integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))\, ds \tag{6}$$

In ADM, y is considered as

$$y = y_0 + \sum_{n=1}^{\infty} y_n \tag{7}$$

And the nonlinear term in

$$f(t, y) = \sum_{n=0}^{\infty} A_n(t, y_0, y_1, y_2, \ldots, y_n) \tag{8}$$

where A_n are the Adomian polynomial. For more on Adomian polynomials and the concept of ADM see Adomian [5]; Agom and Ogunfiditimi [6] and Agom and Ogunfiditimi [7]; Agom, et al. [8] and Agom, et al. [9] and the references there in. Formally,

$$A_n = \frac{1}{n!}\frac{d^n}{d\xi^n} f\left(t, \sum_{i=0}^{\infty} \xi^i y_i\right)_{|\xi=0} \tag{9}$$

where n is a non-negative integer and ξ is a grouping parameter.

3. NUMERICAL ILLUSTRATIONS

In this section we present some numerical results which are adapted from Yakubu, et al. [4]

Example 1

In relation to equations (1) and (2), consider a stiff differential equation with

$$f(t, y) = \frac{y}{4}\left(1 - \frac{y}{20}\right) \tag{10}$$

$y(t_0) = y(0)$, a = 1

The analytical solution is

$$y = \frac{20}{1 + 19e^{-\frac{t}{4}}} \tag{11}$$

which can be expanded in series form to give

$$y = 1 + \frac{19}{80}t + \frac{171}{6400}t^2 + \frac{2717}{1536000}t^3 + \frac{2451}{40960000}t^4 - \frac{11723}{9830400000}t^5 - \frac{78869}{262144000000}t^6 - \ldots \tag{12}$$

Applying equations (2) to (5) on (10), as given in Yakubu, et al. [4] the result by LRKCM is stated in Table 1 with step size of 0.1. From Table 1, the Lagranges interpolation polynomial alongside equation (11) gives the plot in Figure 1A.

Table-1. Exact versus LRKCM values of Example 1

t	Exact value	LRKCM
0.1	1.024018962351866	1.02401896229202
0.2	1.048582996382734	1.04858299626072
0.3	1.073702928838836	1.07370292865234
0.4	1.099389726731483	1.09938972647798
0.5	1.125654495329782	1.12565449500686

Table-2. Exact versus LRKCM values of Example 2.

t	Exact value	LRKCM
0.1	1.024018962351866	1.02401896229202
0.2	1.048582996382734	1.04858299626072
0.3	1.073702928838836	1.07370292865234
0.4	1.099389726731483	1.09938972647798
0.5	1.125654495329782	1.12565449500686

Applying equations (6) to (9) of ADM concept, we have the Adomian polynomial of the nonlinear term as

$$A_0 = y_0^2$$

$$A_1 = 2y_0y_1$$

$$A_2 = 2y_0y_2 + y_1^2$$

. . .

For details see Agom, et al. [8] and Agom and Ogunfiditimi [6] and the references there in. Where

$$y_0 = 1 \tag{13}$$

$$y_1 = \frac{19}{80}t \tag{14}$$

$$y_2 = \frac{171}{6400}t^2 \tag{15}$$

$$y_3 = \frac{2717}{1536000}t^3 \tag{16}$$

and so on. Equations (13) to (16) are the terms of equation (12) which is the exact solution of stiff differential equation (10). The result is presented in Figure 1B with $y = \sum_{n=0}^{7} y_n$

Example 2

Also in relation to equations (1) and (2), consider

$$f(t, y) = -y \tag{17}$$

$$y(t_0) = y(0), \ a = 1$$

The solution is

$$y = e^{-t} \quad (18)$$

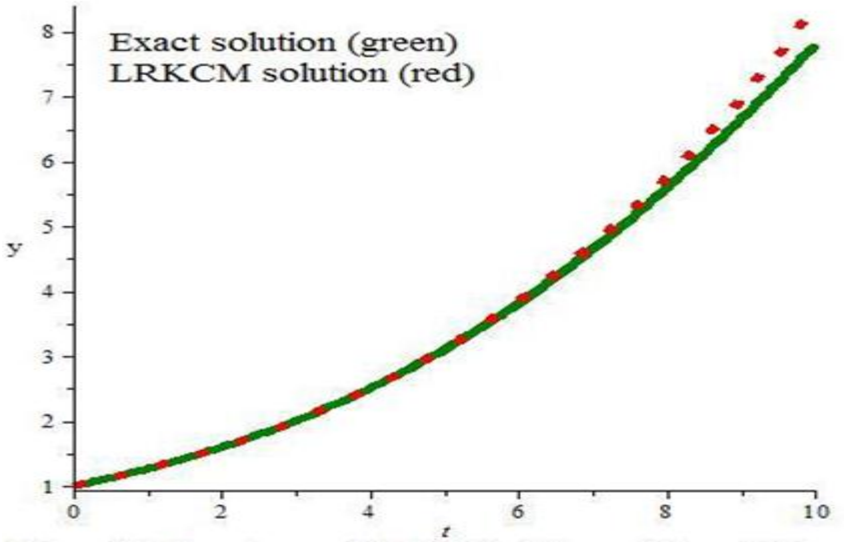

Figure-1A. Exact versus LRKCM Solutions of Example1

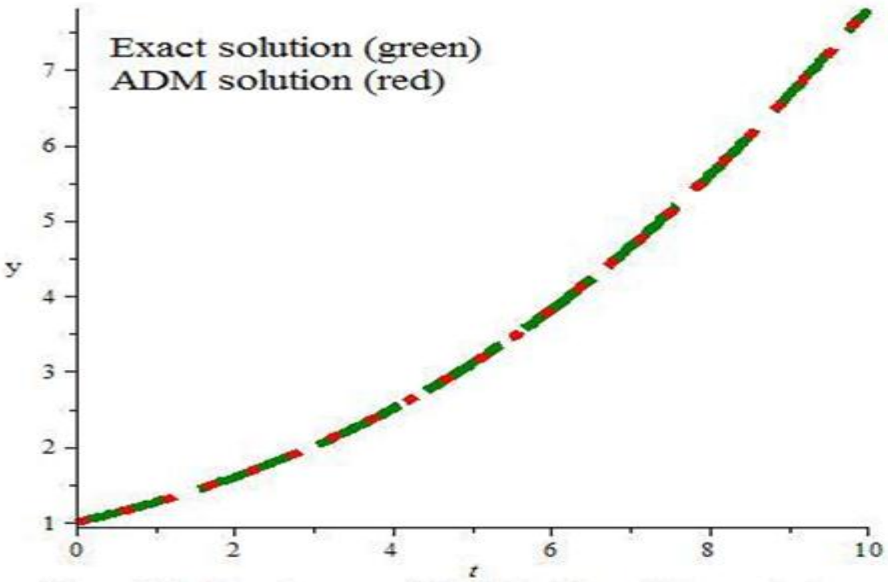

Figure-1B. Exact versus ADM Solutions of Example1

Following similar steps in example 1, we have Table 2 showing solutions by analytical method and LRKCM and the plot is as given in Figure 2A.
Similarly using ADM concept, we have

$$y = \sum_{n=0}^{\infty} y_n = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \quad (19)$$

Equation (19) is the series form of equation (18). The plot of equation (18) and $y = \sum_{n=0}^{7} (-1)^n \frac{t^n}{n!}$ is as given in Figure 2B.

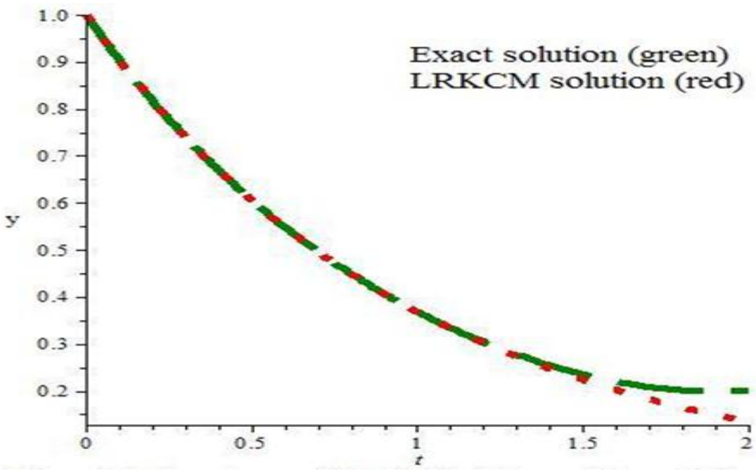

Figure-2A. Exact versus LRKCM Solutions of Example2

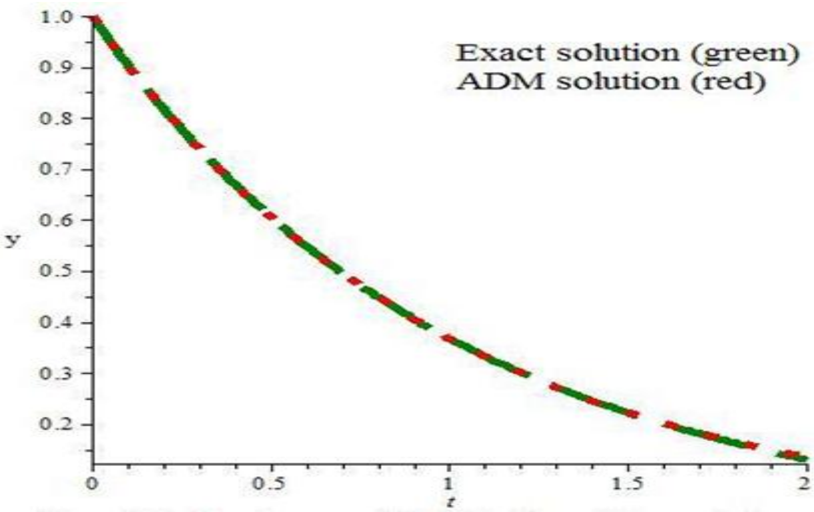

Figure-2B. Exact versus ADM Solutions of Example2

5. CONCLUSION

In this work, we have been able to show that ADM uses no step-size but provided exact solutions to stiff differential equation (1); whereas, LRKCM when applied to the same class of equation provided solutions that were in agreement (to some extend) with the exact solutions and ADM solutions. These discoveries were illustrated in respective plots and tables. Also, it has been reported in literature that LRKCM has been applied to obtain fairly accurate result with extremely small step-size to this class of equations. However, this comes with computational cost unlike ADM.

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