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OPTICAL SOLITARY WAVE SOLUTIONS OF THE SPACE-TIME FRACTIONAL MODIFIED EQUAL-WIDTH EQUATION AND THEIR APPLICATIONS

Dianchen Lu[1]+
Shenghui Ye[2]

[1,2]*Department of Mathematics, Faculty of Science, Jiangsu University, Zhenjiang, China.*
[1]*Email: dclu@ujs.edu.cn*
[2]*Email: 1807392279@qq.com*

(+ Corresponding author)

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ABSTRACT

The space-time fractional modified equal-width equation is a class of fractional partial differential equations which have been used widely in nonlinear optics, solid state physics. In this article, the improved (G'/G)- expansion method has been proposed to construct more new exact solutions of the space-time fractional modified equal-width equation in the sense of modified Riemann-Liouville derivative. The traveling wave transform has been extended to convert the fractional order partial differential equation into an ordinary differential equation. In the end, three families of exact analytical solutions are obtained and expressed them in terms of the hyperbolic, trigonometric, and rational functions with arbitrary parameters, Which reveals that the improved (G'/G) -expansion method is very effective and reliable for solving fractional order partial differential equations. Moreover, the graphical representation of solution is given at different values of α, Which is helpful for people to better study the physical structure of solutions and to analyze the nonlinear optical problems in nonlinear systems.

Contribution/Originality: This study contributes in the existing literature through the improved (G'/G) -expansion method. By applying this method, more new exact solutions of the space-time fractional modified equal-width equation are obtained. The resulting solutions are useful for analyzing nonlinear optics problems.

1. INTRODUCTION

Nonlinear differential equations involving fractional order derivatives are general forms of integer-order classical differential equations. It is well known that nonlinear fractional differential equations (FDEs) cover many fields such as physics, biomechanics, chemistry, biology, power-law non-locality, relativity, nonlinear optics, engineering, solid mechanics, electricity, signal processing and so on Kaplan and Bekir [1]; Sakar and Saldır [2]; Chen and Jiang [3]; Ilie, et al. [4]; Roy, et al. [5]; Ortigueira, et al. [6]; Abrahim [7]; Subasi, et al. [8]; Agom, et al. [9]. These fractional differential equations (FDEs) play a crucial role in almost all areas of life. In order to better understand the motion law of nonlinear phenomena in natural sciences, it is essential to find the traveling wave solutions of FDEs.

In the literature, problems involving nonlinear differential equations and systems are solved by many different powerful methods, such as the extended quantum Zakharov–Kuznetsov equation by Raza, et al. [10] the generalized nonlinear Klein-Gordon equation by Gepreel, et al. [11] the Korteweg-de Vries-Bejamin-Bona-Mahony equation by Simbanefayi and Khalique [12] the generalized Radhakrishnan-Kundu-Lakshmanan dynamical equation with power law nonlinearity by Lu, et al. [13] the nonlinear complex fractional generalized Zakharov dynamical system by Lu, et al. [14] the nonlinear complex fractional Schrodinger equation by Khater, et al. [15] modified KdV–Zakharov–Kuznetsov dynamical equation by Abdullah, et al. [16] the (2+1)-dimensional Boussineq dynamical equation by Ali, et al. [17] coupled Drinfel'd-Sokolov-Wilson equation by Tariq and Seadawy [18] the symmetric regularized long wave equation by Lu, et al. [19].

The nonlinear modified equal width equation is an important mathematical model used for describing various fluid mechanics in nonlinear systems, plasma physics, and nonlinear optics [20]. So far, with the development of symbolic computation software such as Maple, many effective techniques have been proposed to study traveling wave solutions of the nonlinear modified equal width equation. In Pinar and Öziş [21] the solutions of an original auxiliary equation of first-order nonlinear ordinary differential equation with the sixth-degree nonlinear term are given to obtain exact solutions of the modified equal width equation. Application of the dynamical system method is shown in Su and Tang [22] to study the exact travelling wave solutions of the modified equal width equation. Solitary waves of modified equal width equation are presented by direct integration in Yang and Xu [23]. The extended simple equation method and the exp $(-\varphi(\xi))$ expansion method are used for solving the modified equal width equation in Lu, et al. [24]. K.R. Raslan and Khalid K. Ali employ the modified extended tanh method for solving the space-time fractional modified equal width equation in Raslan, et al. [25]. Alper Korkmaz implements various ansatz method to construct solutions of the space-time fractional modified equal width equation [26]. In the present paper, the improved (G'/G)-expansion method is applied to find more new and comprehensive exact solutions of the space–time fractional modified equal-width equation.

The structure of this paper is as follows. In Section 2, the basic definitions and properties of fractional calculus are discussed, and the analysis of the improved (G'/G)-expansion method is formulated in Section 3. The complex exact solutions of the space-time fractional modified equal-width equation are obtained in Section 4. Finally, the conclusions and the advantages of the method are given in Section 5.

2. THE DEFINITION OF THE FRACTIONAL CALCULUS AND ITS THEORY

The Jumarie's modified Riemann–Liouville [27, 28] derivative (RLD) operator of order α for the continuous function f: R $\rightarrow$R is defined by the following expression:

$$D_x^{\alpha} f(x)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x (x-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, 0<\alpha<1\\ (f^{(n)}(x))^{\alpha-n}; n\le\alpha\le n+1, n\ge 1\end{cases}, \tag{1}$$

which will be applied in the following content, where the Gamma function is denoted as

$$\Gamma(\alpha)=\lim_{p\to\infty}\frac{p!\,p^{\alpha}}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+p)}.$$

Some meaningful parts of the modified RLD can be listed as below:

$$D_x^{\alpha} x^{\gamma}=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha}$$

$$D_x^{\alpha}\{af(x)+bg(x)\}=aD_x^{\alpha} f(x)+bD_x^{\alpha} g(x),$$

where a, b are constants and $\gamma \in R$

$$D_x^{\alpha}\{f(x)g(x)\} = g(x)D_x^{\alpha}\{f(x)\} + f(x)D_x^{\alpha}\{g(x)\}.$$

3. DESCRIPTION OF THE IMPROVED (G'/G)-EXPANSION METHOD

In this section, the improved (G'/G)-expansion method [29] has been discussed to obtain the solutions of nonlinear partial differential equations of fractional order. For this, consider the nonlinear fractional partial differential equation in the form:

$$f(u, D_t^{\alpha}u, D_x^{\beta}u, D_{tt}^{2\alpha}u, D_{xx}^{2\beta}u, D_t^{\alpha}D_x^{\beta}u, \cdots) = 0, 0 < \alpha, \beta < 1. \tag{2}$$

Where α and β are fractional orders defined in the sense of the modified Rieman-Liouville derivative and $u = u(x,t)$ is an unknown function, f is a polynomial in u and its fractional derivative involving nonlinear terms and highest derivatives, t is the time variable and x is the space variable. Then the following four steps of the improved (G'/G)-expansion method are given:

Step1: Combine the independent variable x and t into $\eta = \frac{kx^{\beta}}{\Gamma(1+\beta)} - \frac{ct^{\beta}}{\Gamma(1+\beta)}$.We use the traveling wave variable transformation in Equation 2:

$$u(x,t) = u(\eta), \qquad \eta = \frac{kx^{\beta}}{\Gamma(1+\beta)} - \frac{ct^{\beta}}{\Gamma(1+\beta)} \tag{3}$$

where k and c are non-zero constants. By substituting Equation 3 into Equation 2, then Equation 2 is reduced to a nonlinear ordinary differential equation of the polynomial in the form:

$$F(u, u', u'', u''', \cdots) = 0 \tag{4}$$

Where F is a function of $u = u(\eta)$ and its derivatives with respect to η.

Step2: Suppose the traveling wave solution of Equation 4 can be expressed in the following form:

$$u(\eta) = \sum_{i=0}^{m} \alpha_i (G'/G)^i + \sum_{i=1}^{m} \beta_i (G'/G)^{-i} \tag{5}$$

where either α_m or β_m may be zero, but both of them cannot be zero simultaneously such that $\alpha_i (i = 0,1,2,\ldots m), \beta_i (i = 1,2,\ldots,m)$ are arbitrary constants to be determined later [30]. and $G = G(\xi)$ satisfies the second-order nonlinear ordinary differential equation

$$GG'' = \lambda GG' + \mu G^2 + \nu (G')^2 \tag{6}$$

Where the derivative denoted by the prime with respect to η, λ, μ and ν are genuine constants. By using Cole-Hopf transformation $\phi(\eta) = \ln(G(\eta))_\eta = (G'(\eta)/G(\eta))$ simplify Equation 6 into the Riccati equation

$$\phi'(\eta) = \mu + \lambda\phi(\eta) + (\nu - 1)\phi^2(\eta) \tag{7}$$

The generalized Riccati Equation 7 has twenty-five distinct solutions [31] and (see Appendix 1 for details)

Step3: Balancing the higher order derivative term and the higher order nonlinear term of (4) consequently, the value of the positive integer m can be obtained. The detailed expression is the same as the following formula

$$D[u(\eta)] = n; D[\frac{d^m u(\eta)}{d\eta^m}] = m + n; D[(u^m \frac{d^K u(\eta)}{d\eta^K})^S] = nm + S(n+K). \tag{8}$$

Step4: By Substituting Equation 5 into Equation 4 and using Equation 7, we get polynomials in $(G'/G)^i (i = 0,1,\ldots,m)$ and $(G'/G)^{-i}(i = 1,2,\ldots,m)$. Then equating each coefficient of the resulting polynomials to zero, yields a system of over-determined equations for $\alpha_0, \alpha_1, \ldots \alpha_m, \beta_1, \ldots \beta_m, c$ and ν.

Step5: Solving the algebraic equations system in Step 4 and obtaining the unknown constants. Substituting values of the constants together with the solutions of Equation 7, new abundant and general type exact traveling wave solutions will be obtained for Equation 2.

4. THE APPLICATION OF THE METHOD

In this section, we consider space-time fractional modified equal-width equation [25, 26]:

$$D_t^\beta u(x,t) + \varepsilon D_x^\beta u^3(x,t) - \delta D_{xxt}^{3\beta} u(x,t) = 0. \tag{9}$$

Where ε and δ are real parameters. Utilize the complex transformation for fractional differential Equation 3 of Step 1:

$$u(x,t) = u(\eta), \eta = \frac{kx^\beta}{\Gamma(1+\beta)} - \frac{ct^\beta}{\Gamma(1+\beta)} \tag{10}$$

Then Equation 9 can be reduced into the following nonlinear ordinary differential equation:

$$-cu' + \varepsilon k(u^3)' + \delta ck^2 u''' = 0 \tag{11}$$

Integrating Equation 11 once with respect to η, we obtain

$$-cu + \varepsilon ku^3 + \delta ck^2 u'' = 0 \tag{12}$$

According to the third step of the improved (G'/G)-expansion method, using the balancing principle between u'' and u^3 in Equation 12 gives $m = 1$. Therefore, Equation 5 can be written as:

$$u(\xi) = \alpha_0 + \alpha_1(\frac{G'}{G}) + \beta_1(\frac{G'}{G})^{-1}, \alpha_1 \neq 0, or \beta_1 \neq 0 \tag{13}$$

where α_0, α_1 and β_1 are constants to be determined later. Inserting Equation 13 and its derivative with Equation 6 and Equation 7 into Equation 12, the left-hand side is transformed into polynomials in

$(G'/G)^i (i=0,1,2,\ldots,m)$ and $(G'/G)^{-i} (i=1,2,3,\ldots,m)$. Assembling each coefficient of the resulting polynomials to zero. Subsequently, we attain a set of algebraic equations for $\alpha_0, \alpha_1, \beta_1, \nu$ and c (which we do not include it here for simplicity). To obtain the following two different sets of values, we can solve the over-determined system of algebraic equations by any computer program like Maple, Mathematica.

Case-1:

$$\alpha_0 = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}, \alpha_1 = \mp i \frac{(2\sqrt{2c} + k^2\delta\lambda^2\sqrt{2c})}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}, \beta_1 = 0, \nu = \frac{2 + k^2\delta\lambda^2 + 4k^2\delta\mu}{4k^2\delta\mu} \tag{14}$$

Case-2:

$$\alpha_0 = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}, \alpha_1 = 0, \beta_1 = \mp i\mu \frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}, \nu = \frac{2 + k^2\delta\lambda^2 + 4k^2\delta\mu}{4k^2\delta\mu} \tag{15}$$

Substituting Equation 14, Equation 15 into Equation 13, we obtain respectively:

$$u_1(x,t) = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i \frac{\left(2\sqrt{2c} + k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \left(\frac{G'}{G}\right) \tag{16}$$

$$u_2(x,t) = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i\mu \frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \left(\frac{G'}{G}\right)^{-1} \tag{17}$$

Inserting the solutions of the Equation 7 see Appendix 1 into Equation 16 and simplify, we achieve the following solutions.

When $\Omega = \lambda^2 - 4\mu(\nu-1) > 0$ and $\lambda(\nu-1) \neq 0 \ \ (or\, \mu(\nu-1) \neq 0)$

$$u_{1_1}(x,t) = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i \frac{\left(2\sqrt{2c} + k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot \left\{ -\frac{1}{2(\nu-1)} \left[\lambda + \sqrt{\Omega} \tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right) \right] \right\} \tag{18}$$

$$u_{1_2}(x,t) = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i \frac{\left(2\sqrt{2c} + k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot \left\{ -\frac{1}{2(\nu-1)} \left[\lambda + \sqrt{\Omega} \coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right) \right] \right\} \tag{19}$$

$$u_{1_3}(x,t) = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i \frac{\left(2\sqrt{2c} + k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot \left\{ -\frac{1}{2(\nu-1)} \left[\lambda + \sqrt{\Omega}\left(\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right) \pm i\,\mathrm{sech}\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right) \right] \right\} \tag{20}$$

$$u_{1_4}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}$$
$$\cdot\left\{-\frac{1}{2(\nu-1)}\left[\lambda+\sqrt{\Omega}(\coth(\frac{1}{2}\sqrt{\Omega}\eta)\pm\operatorname{csch}(\frac{1}{2}\sqrt{\Omega}\eta))\right]\right\} \tag{21}$$

$$u_{1_5}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}$$
$$\cdot\left\{-\frac{1}{4(\nu-1)}\left[2\lambda+\sqrt{\Omega}(\tanh(\frac{1}{4}\sqrt{\Omega}\eta)\pm\coth(\frac{1}{4}\sqrt{\Omega}\eta))\right]\right\} \tag{22}$$

$$u_{1_6}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}$$
$$\cdot\left\{\frac{1}{2(\nu-1)}\left[-\lambda+\frac{\pm\sqrt{\Omega\left(A^2+B^2\right)}-A\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\eta\right)}{A\sinh\sqrt{\Omega}\eta+B}\right]\right\} \tag{23}$$

And the plot of u_{1_6} for $k=c=\lambda=1,\mu=-1,\nu=2,\delta=2,\varepsilon=-1$ and $\alpha=0.25,0.5,0.75,1$ is displayed in Figure1 (a)-(d). Consider

$$u_{1_7}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}$$
$$\cdot\left\{\frac{1}{2(\nu-1)}\left[-\lambda-\frac{\pm\sqrt{\Omega\left(A^2+B^2\right)}+A\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\eta\right)}{A\sinh\sqrt{\Omega}\eta+B}\right]\right\} \tag{24}$$

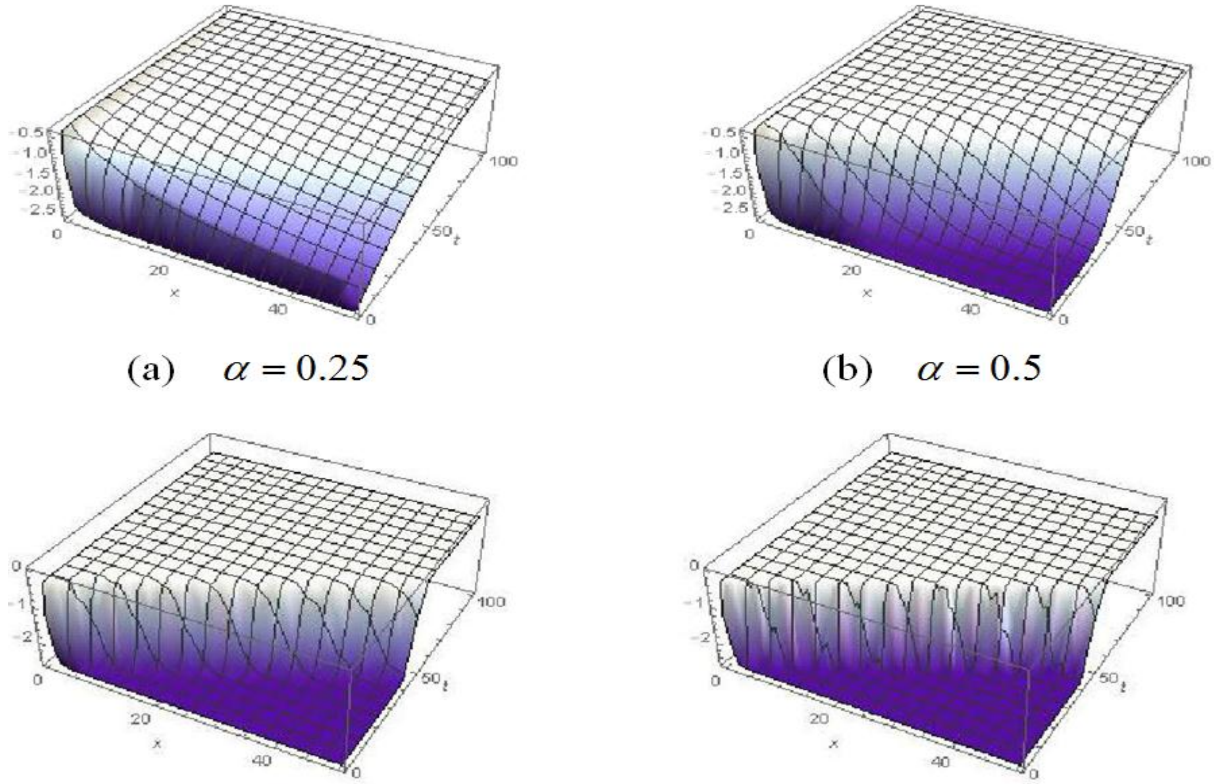
Where A and B are non-zero constants. Consider the following:

$$u_{1_8}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}$$
$$\cdot\left(\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\right)+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)}\right) \tag{25}$$

$$u_{1_9}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}$$
$$\cdot\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\right)-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)}\right) \tag{26}$$

$$u_{1_{10}}(x,t) = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i \frac{\left(2\sqrt{2c} + k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot \left(\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\right) + \lambda\cosh\left((1/2)\sqrt{\Omega}\right) \pm i\sqrt{\Omega}} \right) \tag{27}$$

$$u_{1_{11}}(x,t) = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i \frac{\left(2\sqrt{2c} + k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot \left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\right) - \lambda\sinh\left((1/2)\sqrt{\Omega}\right) \pm \sqrt{\Omega}} \right) \tag{28}$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.5$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

Figure-1. (a)- (d) show the dark solitary wave solutions for u_{1_6} at different values of α.

Source: The figure is plotted by Mathematica.

When $\Omega = \lambda^2 - 4\mu(\nu - 1) < 0$ and $\lambda(\nu - 1) \neq 0$ (or $\mu(\nu - 1) \neq 0$)

$$u_{1_{12}}(x,t) = \mp i\lambda \frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i \frac{\left(2\sqrt{2c} + k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot \left\{ \frac{1}{2(\nu-1)} \left[-\lambda + \sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right) \right] \right\} \tag{29}$$

$$u_{1_{13}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}\cdot\left\{\frac{1}{2(\nu\text{-}1)}\left(\text{-}\lambda+\frac{\pm\sqrt{\Omega\left(A^2+B^2\right)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh\sqrt{\Omega}\eta+B}\right)\right\} \tag{30}$$

$$u_{1_{14}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}\cdot\left\{\frac{1}{2(\nu\text{-}1)}\left[\text{-}\lambda+\sqrt{\text{-}\Omega}(\tan(\frac{1}{2}\sqrt{\text{-}\Omega}\eta)\pm\sec\left(\frac{1}{2}\sqrt{-\Omega}\eta\right))\right]\right\} \tag{31}$$

$$u_{1_{15}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}\cdot\left\{-\frac{1}{2(\nu\text{-}1)}\left[\lambda+\sqrt{\text{-}\Omega}(\cot(\frac{1}{2}\sqrt{\text{-}\Omega}\eta)\pm\csc\left(\frac{1}{2}\sqrt{-\Omega}\eta\right))\right]\right\} \tag{32}$$

$$u_{1_{16}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}\cdot\left\{\frac{1}{4(\nu\text{-}1)}\left[\text{-}2\lambda+\sqrt{\text{-}\Omega}(\tan(\frac{1}{4}\sqrt{\text{-}\Omega}\eta)+\cot\left(\frac{1}{4}\sqrt{-\Omega}\eta\right))\right]\right\} \tag{33}$$

$$u_{1_{17}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}\cdot\left\{\frac{1}{2(\nu\text{-}1)}\left(\text{-}\lambda+\frac{\pm\sqrt{\text{-}\Omega\left(A^2\text{-}B^2\right)}-A\sqrt{\text{-}\Omega}\cos(\sqrt{\text{-}\Omega}\eta)}{A\sin\sqrt{\text{-}\Omega}\eta+B}\right)\right\} \tag{34}$$

$$u_{1_{18}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}\cdot\left\{\frac{1}{2(\nu\text{-}1)}\left(\text{-}\lambda\text{-}\frac{\pm\sqrt{\text{-}\Omega\left(A^2\text{-}B^2\right)}+A\sqrt{\text{-}\Omega}\cos(\sqrt{\text{-}\Omega}\eta)}{A\sin\sqrt{\text{-}\Omega}\eta+B}\right)\right\} \tag{35}$$

Where A and B are non-zero constants such that $A^2-B^2>0$. Consider the following:

$$u_{1_{19}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot\left(\frac{-2\mu\sqrt{-\Omega}\cos\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\right)+\lambda\cos\left((1/2)\sqrt{-\Omega}\right)}\right) \tag{36}$$

$$u_{1_{20}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot\left(\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\right)-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)}\right) \tag{37}$$

$$u_{1_{21}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot\left(\frac{-2\mu\sqrt{-\Omega}\cos\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\right)+\lambda\cos\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}\right) \tag{38}$$

$$u_{1_{22}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot\left(\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\right)-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}\right) \tag{39}$$

When $\mu=0$ and $\lambda(\nu-1)\neq 0$

$$u_{1_{23}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot\left(\frac{\lambda k}{(\nu-1)\left(k+\cosh(\lambda\eta)-\lambda\sinh(\lambda\eta)\right)}\right) \tag{40}$$

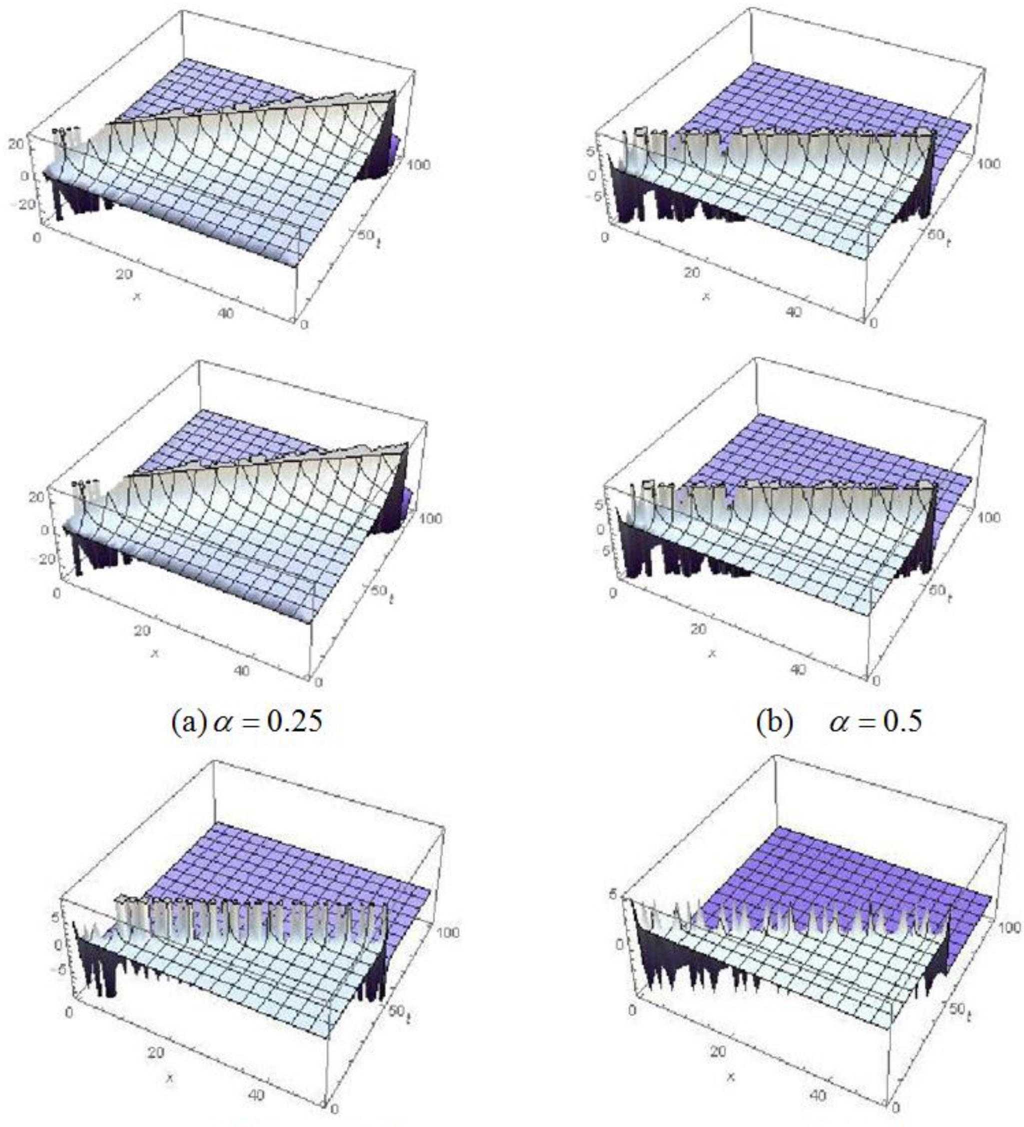
$$u_{1_{24}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}} \cdot\left(\frac{\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta)}{(\nu-1)\left(k+\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta)\right)}\right) \tag{41}$$

Where k is an arbitrary constant.

When $\mu=\lambda=0$ and $(\nu-1)\neq 0$

$$u_{1_{25}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\frac{\left(2\sqrt{2c}+k^2\delta\lambda^2\sqrt{2c}\right)}{4k^{3/2}\mu\sqrt{\delta\varepsilon}}\left(-\frac{1}{(\nu-1)\eta+c_1}\right) \tag{42}$$

Where c_1 is an arbitrary constant. Substituting the solutions of the Equation 7 (see Appendix 1) into Equation 17 and simplify. We obtain the following solutions. when $\Omega=\lambda^2\text{-}4\mu(\nu-1)>0$ and $\lambda(\nu\text{-}1)\neq 0$ (or $\mu(\nu-1)\neq 0$)



(a) $\alpha=0.25$ (b) $\alpha=0.5$

(c) $\alpha=0.75$ (d) $\alpha=1$

Figure-2. (a)-(d) shows the solitary wave solutions for u_{2_1} at different values of α .

Source: The figure is plotted by Mathematica.

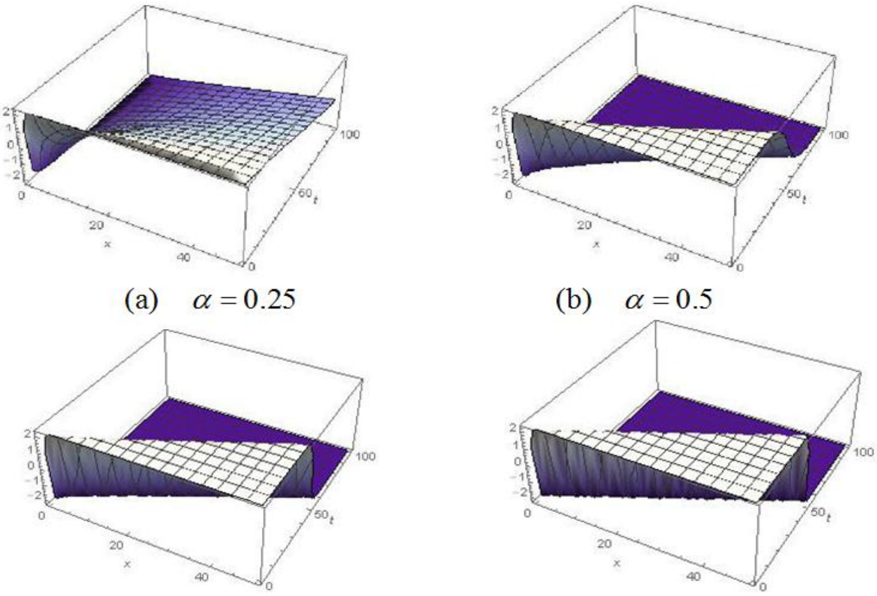
$$u_{2_1}(x,t) = \mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot \left\{-\frac{1}{2(\nu-1)}\left[\lambda+\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right]\right\}^{-1} \tag{43}$$

and the plot of u_{2_1} for $k=c=\lambda=1, \mu=-1, \nu=2, \delta=-2, \varepsilon=1$ and $\alpha=0.25, 0.5, 0.75, 1$ is displayed in Figure 2(a)-(d). Consider

$$u_{2_2}(x,t) = \mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot \left\{-\frac{1}{2(\nu-1)}\left[\lambda+\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right]\right\}^{-1} \tag{44}$$

and the plot of u_{2_2} for $k=c=\lambda=1, \mu=-1, \nu=2, \delta=-2, \varepsilon=1$ and $\alpha=0.25, 0.5, 0.75, 1$ is displayed in Figure 3(a)-(d). Consider

$$u_{2_3}(x,t) = \mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot \left\{-\frac{1}{2(\nu-1)}\left[\lambda+\sqrt{\Omega}\left(\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right) \pm i\,\mathrm{sech}\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)\right]\right\}^{-1} \tag{45}$$



(a) $\alpha=0.25$ (b) $\alpha=0.5$

(c) $\alpha=0.75$ (d) $\alpha=1$

Figure-3. (a)-(d) shows the dark solitary wave solutions for u_{2_2} at different values of α.

Source: The figure is plotted by Mathematica.

$$u_{2_4}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}$$
$$\cdot\left\{-\frac{1}{2(\nu-1)}\left[\lambda+\sqrt{\Omega}\left(\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\pm\operatorname{csch}\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)\right]\right\}^{-1} \quad (46)$$

$$u_{2_5}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}$$
$$\cdot\left\{-\frac{1}{4(\nu-1)}\left[2\lambda+\sqrt{\Omega}\left(\tanh\left(\frac{1}{4}\sqrt{\Omega}\eta\right)+\coth\left(\frac{1}{4}\sqrt{\Omega}\eta\right)\right)\right]\right\}^{-1} \quad (47)$$

$$u_{2_6}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}$$
$$\cdot\left\{\frac{1}{2(\nu-1)}\left(-\lambda+\frac{\pm\sqrt{\Omega\left(A^2+B^2\right)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh\sqrt{\Omega}\eta+B}\right)\right\}^{-1} \quad (48)$$

$$u_{2_7}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}$$
$$\cdot\left\{\frac{1}{2(\nu-1)}\left(-\lambda-\frac{\pm\sqrt{\Omega\left(A^2+B^2\right)}+A\sqrt{\Omega}\cosh(\sqrt{\Omega}\eta)}{A\sinh\sqrt{\Omega}\eta+B}\right)\right\}^{-1} \quad (49)$$

Where A and B are non-zero constants. Consider the following:

$$u_{2_8}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\left(\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\right)+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)}\right)^{-1} \quad (50)$$

$$u_{2_9}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}$$
$$\cdot\left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\right)-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)}\right)^{-1} \quad (51)$$

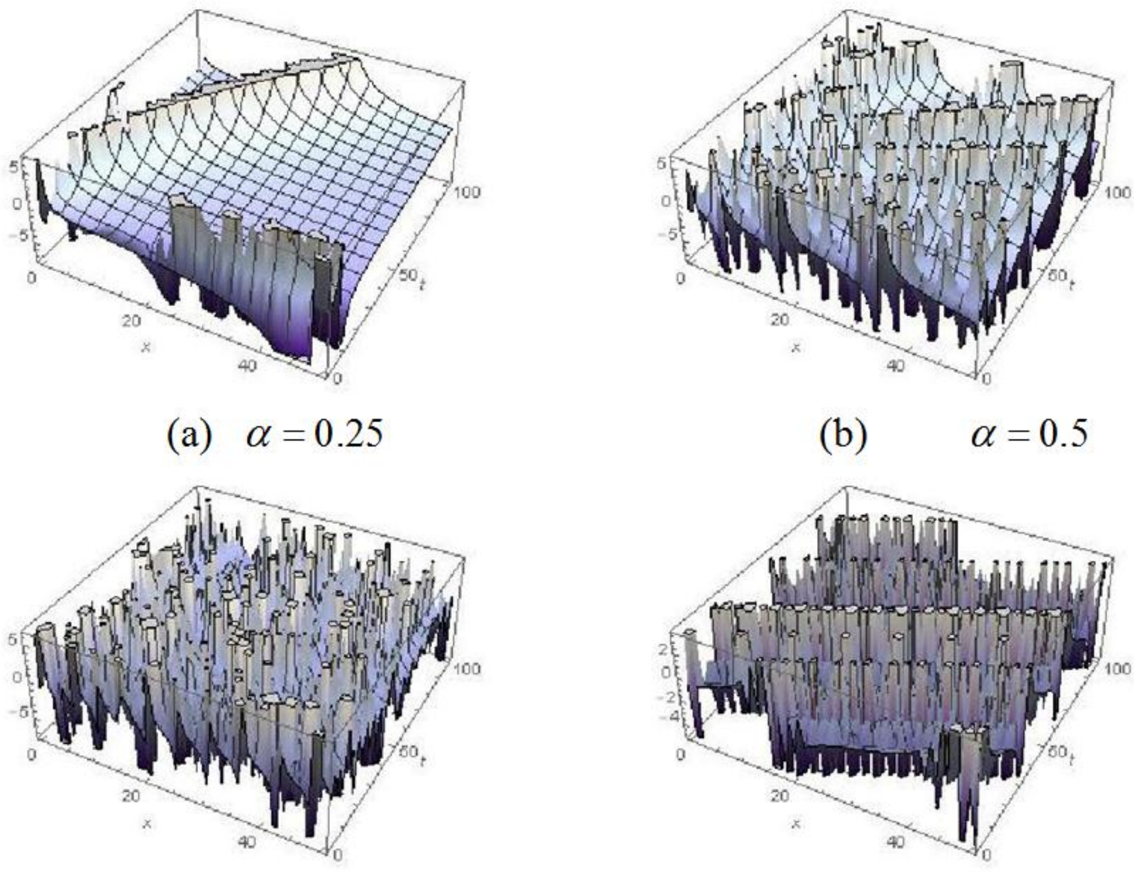
$$u_{2_{10}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}$$
$$\cdot\left(\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\right)+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)\pm i\sqrt{\Omega}}\right)^{-1} \quad (52)$$

$$u_{2_{11}}(x,t) = \mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot \left(\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\right)-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)\pm\sqrt{\Omega}}\right)^{-1} \quad (53)$$

When $\Omega = \lambda^2 - 4\mu(\nu-1) < 0$ and $\lambda(\nu-1) \neq 0$ or $\mu(\nu-1) \neq 0$

$$u_{2_{12}}(x,t) = \mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}} \mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot \left\{\frac{1}{2(\nu-1)}\left[-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right]\right\}^{-1} \quad (54)$$

and the plot of $u_{2_{12}}$ for $k=c=\lambda=1, \mu=-1, \nu=-2, \delta=-2, \varepsilon=1$ and $\alpha=0.25, 0.5, 0.75, 1$ is displayed in Figure 4(a)-(d). Consider



(a) $\alpha = 0.25$ (b) $\alpha = 0.5$ (b) $\alpha = 0.75$ (d) $\alpha = 1$

Figure-4. (a)-(d) Show the periodic solitary wave solutions for $u_{2_{12}}$ at different values of α.

Source: The figure is plotted by Mathematica.

$$u_{2_{13}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\cdot\left\{\frac{1}{2(\nu\text{-}1)}\left[-\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right]\right\}^{-1} \tag{55}$$

$$u_{2_{14}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\cdot\left\{\frac{1}{2(\nu\text{-}1)}\left[-\lambda+\sqrt{-\Omega}\left(\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\pm\sec\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)\right]\right\}^{-1} \tag{56}$$

$$u_{2_{15}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\cdot\left\{-\frac{1}{2(\nu\text{-}1)}\left[\lambda+\sqrt{-\Omega}\left(\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\pm\csc\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)\right]\right\}^{-1} \tag{57}$$

$$u_{2_{16}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\cdot\left\{\frac{1}{4(\nu\text{-}1)}\left[-2\lambda+\sqrt{-\Omega}\left(\tan\left(\frac{1}{4}\sqrt{-\Omega}\eta\right)+\cot\left(\frac{1}{4}\sqrt{-\Omega}\eta\right)\right)\right]\right\}^{-1} \tag{58}$$

$$u_{2_{17}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\cdot\left\{\frac{1}{2(\nu\text{-}1)}\left(-\lambda+\frac{\pm\sqrt{-\Omega\left(A^2\text{-}B^2\right)}-A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\eta)}{A\sin\sqrt{-\Omega}\eta+B}\right)\right\}^{-1} \tag{59}$$

$$u_{2_{18}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\cdot\left\{\frac{1}{2(\nu\text{-}1)}\left(-\lambda-\frac{\pm\sqrt{-\Omega\left(A^2\text{-}B^2\right)}+A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\eta)}{A\sin\sqrt{-\Omega}\eta+B}\right)\right\}^{-1} \tag{60}$$

Where A and B are non-zero constants such that $A^2-B^2>0$. Consider the following

$$u_{2_{19}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\cdot\left(\frac{-2\mu\sqrt{-\Omega}\cos\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\right)+\lambda\cos\left((1/2)\sqrt{-\Omega}\right)}\right)^{-1} \tag{61}$$

$$u_{2_{20}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot\left(\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\right)-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)}\right)^{-1} \tag{62}$$

$$u_{2_{21}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot\left(\frac{-2\mu\sqrt{-\Omega}\cos\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\right)+\lambda\cos\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}\right)^{-1} \tag{63}$$

$$u_{2_{22}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot\left(\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\right)-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}\right)^{-1} \tag{64}$$

When $\mu=0$, and $\lambda(\nu-1)\neq 0$

$$u_{2_{23}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot\left(\frac{\lambda k}{(\nu-1)\left(k+\cosh(\lambda\eta)-\lambda\sinh(\lambda\eta)\right)}\right)^{-1} \tag{65}$$

$$u_{2_{24}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}} \cdot\left(\frac{\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta)}{(\nu-1)\left(k+\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta)\right)}\right)^{-1} \tag{66}$$

Where k is an arbitrary constant.

When $\mu=\lambda=0$, and $\lambda(\nu-1)\neq 0$

$$u_{2_{25}}(x,t)=\mp i\lambda\frac{\sqrt{ck\delta}}{\sqrt{2\varepsilon}}\mp i\mu\frac{\sqrt{2ck\delta}}{\sqrt{\varepsilon}}\cdot\left(-\frac{1}{(\nu-1)\eta+c_1}\right)^{-1} \tag{67}$$

where c_1 is an arbitrary constant.

5. DISCUSSION

From the above examples, we can see that many new and complex exact solutions of the space-time fractional modified equal width equation can be obtained by using the improved (G' /G)-expansion method. The modified extended tanh method is employed to solve the space-time fractional modified equal width equation in Raslan, et al. [25]. As a result, the solution is expressed in the form of hyperbolic function and triangular function. Comparing with the solution in this paper, if the parameters take specific values, the results (18), (19), (22), (29), (33), (43), (44), (47), (54), (55), (58) are consistent with the solutions in Raslan, et al. [25]. In addition, solutions (20), (21), (23)~(28), (30)~(32), (34)~(42), (45), (46), (48)~(53), (56), (57), (59)~(67) are new exact solutions of the space-time fractional modified equal width equation.

Alper Korkmaz use various ansatz method to solve the space-time fractional modified equal width equation [26]. The bright soliton solutions and singular solutions are expressed in the form of hyperbolic functions. However, the solutions (18)~(67) in this paper are obviously different from the results in the literature [26]. In addition, the solutions (29), (31)~(39), (54)~(64) obtained in this paper are in the form of trigonometric functions. The solutions expressed by (42), (67) are rational functions, which can be seen as new solutions obtained by improved (G' /G)-expansion method.

Through the above comparative analysis, the validity, accuracy, superiority and wide applicability of the method are illustrated. It can be extended to different types of fractional partial differential equations.

6. CONCLUSION

In this paper, the improved (G' /G)-expansion method has been employed successfully to obtain the exact solutions of the space-time fractional modified equal-width equation, in the sense of modified Riemann-Liouville derivative. First, we convert the space-time fractional modified equal-width equation into an ordinary differential equation by the fractional complex transformation. As a result, we get new and more general exact solutions which are not found in the previous literature [25, 26]. The obtained exact solutions are reported in terms of the hyperbolic, the trigonometric and the rational functions with some parameters. All the solutions presented in this paper have been checked with Mathematica by putting them back into the original Equation 9. In addition, it is obvious that the method would be a reliable and effective mathematical tool to handle other fractional differential equations from natural sciences.

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Appendix-1

When $\Omega=\lambda^2-4\mu(\nu-1)>0$ and $\lambda(\nu-1)\neq 0$ (or $\mu(\nu-1)\neq 0$)

$$u_1(x,t)=-\frac{1}{2(\nu-1)}\left(\lambda+\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)$$

$$u_2(x,t)=-\frac{1}{2(\nu-1)}\left(\lambda+\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)$$

$$u_3(x,t)=-\frac{1}{2(\nu-1)}\left[\lambda+\sqrt{\Omega}\left(\tanh\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\pm i\,\mathrm{sec}h\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)\right]$$

$$u_4(x,t)=-\frac{1}{2(\nu-1)}\left[\lambda+\sqrt{\Omega}\left(\coth\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\pm \mathrm{csc}h\left(\frac{1}{2}\sqrt{\Omega}\eta\right)\right)\right]$$

$$u_5(x,t)=-\frac{1}{4(\nu-1)}\left[2\lambda+\sqrt{\Omega}\left(\tanh\left(\frac{1}{4}\sqrt{\Omega}\eta\right)+\coth\left(\frac{1}{4}\sqrt{\Omega}\eta\right)\right)\right]$$

$$u_6(x,t)=\frac{1}{2(\nu-1)}\left(-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\eta\right)}{A\sinh\sqrt{\Omega}\eta+B}\right)$$

$$u_7(x,t)=\frac{1}{2(\nu-1)}\left(-\lambda-\frac{\pm\sqrt{\Omega(A^2+B^2)}+A\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\eta\right)}{A\sinh\sqrt{\Omega}\eta+B}\right)$$

Where A and B are non-zero constants

$$u_8(x,t)=\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\right)+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)}$$

$$u_9(x,t)=\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\right)-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)}$$

$$u_{10}(x,t)=\frac{-2\mu\sqrt{\Omega}\cosh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\right)+\lambda\cosh\left((1/2)\sqrt{\Omega}\right)\pm i\sqrt{\Omega}}$$

$$u_{11}(x,t)=\frac{2\mu\sqrt{\Omega}\sinh\left((1/2)\sqrt{\Omega}\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\right)-\lambda\sinh\left((1/2)\sqrt{\Omega}\right)\pm\sqrt{\Omega}}$$

When $\Omega=\lambda^2-4\mu(\nu-1)<0$ and $\lambda(\nu-1)\neq 0$ (or $\mu(\nu-1)\neq 0$)

$$u_{12}(x,t)=\frac{1}{2(\nu-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)$$

$$u_{13}(x,t)=\frac{1}{2(\nu-1)}\left(-\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)$$

$$u_{14}(x,t)=\frac{1}{2(\nu-1)}\left[-\lambda+\sqrt{-\Omega}\left(\tan\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\pm\sec\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)\right]$$

$$u_{15}(x,t)=\frac{1}{2(\nu-1)}\left[-\lambda+\sqrt{-\Omega}\left(\cot\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\pm\csc\left(\frac{1}{2}\sqrt{-\Omega}\eta\right)\right)\right]$$

$$u_{16}(x,t)=\frac{1}{4(\nu-1)}\left[-2\lambda+\sqrt{-\Omega}\left(\tan\left(\frac{1}{4}\sqrt{-\Omega}\eta\right)+\cot\left(\frac{1}{4}\sqrt{-\Omega}\eta\right)\right)\right]$$

$$u_{17}(x,t)=\frac{1}{2(\nu-1)}\left(-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}-A\sqrt{-\Omega}\cos\sqrt{-\Omega}\eta}{A\sin\sqrt{-\Omega}\eta+B}\right)$$

$$u_{18}(x,t)=\frac{1}{2(\nu-1)}\left(-\lambda-\frac{\pm\sqrt{-\Omega(A^2-B^2)}+A\sqrt{-\Omega}\cos\sqrt{-\Omega}\eta}{A\sin\sqrt{-\Omega}\eta+B}\right)$$

Where A and B are non-zero constants and satisfy the condition $A^2 - B^2 > 0$

$$u_{19}(x,t)=\frac{-2\mu\sqrt{-\Omega}\cos\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\right)+\lambda\cos\left((1/2)\sqrt{-\Omega}\right)}$$

$$u_{20}(x,t)=\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\right)-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)}$$

$$u_{21}(x,t)=\frac{-2\mu\sqrt{-\Omega}\cos\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\right)+\lambda\cos\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}$$

$$u_{22}(x,t)=\frac{2\mu\sqrt{-\Omega}\sin\left((1/2)\sqrt{-\Omega}\right)}{\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\right)-\lambda\sin\left((1/2)\sqrt{-\Omega}\right)\pm\sqrt{-\Omega}}$$

When $\mu=0$ and $\lambda(\nu-1)\neq 0$

$$u_{23}(x,t)=\frac{\lambda k}{(\nu-1)(k+\cosh(\lambda\eta)-\lambda\sinh(\lambda\eta))}$$

$$u_{24}(x,t)=\frac{\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta)}{(\nu-1)(k+\cosh(\lambda\eta)+\lambda\sinh(\lambda\eta))}$$

Where k is an arbitrary constant.

When $\mu=0$ and $\nu-1\neq 0$

$$u_{25}(x,t)=-\frac{1}{(\nu-1)\eta+c_1}$$