International Journal of Mathematical Research
2020 Vol. 9, No. 1, pp. 1-10
ISSN(e): 2306-2223
ISSN(p): 2311-7427
DOI: 10.18488/journal.24.2020.91.1.10

NUMERICAL SOLUTIONS OF BLACK-SCHOLES MODEL BY DU FORT-FRANKEL FDM AND GALERKIN WRM

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Article History
Received: 27 November 2019
Revised: 30 December 2019
Accepted: 3 February 2020
Published: 25 February 2020

Keywords
Black-Scholes equation
European call option
European put option
Du Fort-Frankel finite difference method (DF3DM)
Galerkin weighted residual method (GWRM)
Modified legendre polynomials.

ABSTRACT

The main objective of this paper is to find the approximate solutions of the Black-Scholes (BS) model by two numerical techniques, namely, Du Fort-Frankel finite difference method (DF3DM), and Galerkin weighted residual method (GWRM) for both (call and put) type of European options. Since both DF3DM and GWRM are the most familiar numerical techniques for solving partial differential equations (PDE) of parabolic type, we estimate options prices by using these techniques. For this, we first convert the Black-Scholes model into a modified parabolic PDE, more specifically, in DF3DM, the first temporal vector is calculated by the Crank-Nicolson method using the initial boundary conditions and then the option price is evaluated. On the other hand, in GWRM, we use piecewise modified Legendre polynomials as the basis functions of GWRM which satisfy the homogeneous form of the boundary conditions. We may observe that the results obtained by the present methods converge fast to the exact solutions. In some cases, the present methods give more accurate results than the earlier results obtained by the adomian decomposition method [14]. Finally, all approximate solutions are shown by the graphical and tabular representations.

Contribution/Originality: The paper's primary contribution is finding that the approximate results of Black-Scholes model by DF3DM, and GWRM with modified Legendre polynomials as basis functions.

1. INTRODUCTION

Options are treated as the most important part of the security markets from the beginning of the Chicago Board Options Exchange (CBOE) in 1973, which is the largest options market in the world [1]. During last decades, the valuation of options has become important problem for both financial and mathematical point of view. Details about options are available in Hull [1]; Privault [2]. There are many models for calculating the value of options but among all of those models, the Black-Scholes model is a suitable way to find the European options price.

The discovery of the Black-Scholes model took long time. Fishers Black took the first step to make a model for valuation of stock. Afterward Myron Scholes added with Black and today we use their result for finding the value of different kinds of stocks. In 1973, the concept of the Black-Scholes model was first disclosed in the paper entitled, "The pricing of options and corporate liabilities" in the Journal of political economy by Black and Scholes [3] and then advanced in "Theory of rational option pricing" by Robert Merton. In 2003, Chawla, et al. [4] approximate European put option value by using Generalized trapezoidal formula and found better approximation than Crank-Nicolson method especially near the strike price. In Hackmann [5] Crank-Nicolson method has been used for evaluation of European options price with accuracy up to 3 decimal places. In 2012, Shinde and Takale [6] have calculated European call option values by using Black-Scholes formula. In Jódar, et al. [7] Black-Scholes model was solved by using Mellin transformation without numerical experiment. In Boyle [8] Monte Carlo method has been used for calculating the value of options with the accuracy and reliability. Very recently, Hok and Chan [9] develop a European option pricing method by using Fourier series with Legendre polynomials. In Carr and Madan [10] Peter Carr and Dilip B. Madan have shown the fast Fourier transform with the values of options.

Let us consider, $F(S,t)$ = option price, S = stock price, K = strike price, σ = volatility, T = maturity time, r = risk-free interest rate and t = time in years. Then the famous Black-Scholes PDE is

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0, \qquad 0 < S < \infty,\ 0 < t < T \tag{1}$$

1.1. Terminal and Boundary Conditions

1.1.1. Conditions for European Call Options

Terminal condition: $F(S,t) = max(S-K,0)\ \ at\ t = T$

Boundary conditions: $F(S,t) = 0$, when $S = 0$ and $F(S,t) = S - Ke^{-r(T-t)}$, when $S \to \infty$

1.1.2. Conditions European Put Options

Terminal condition: $F(S,t) = \max(K-S,0)\ \ at\ t = T$

Boundary conditions: $F(S,t) = Ke^{-r(T-t)}$, when $S = 0$ and $F(S,t) = 0$, when $S \to \infty$

1.2. Black-Scholes Option Pricing Formula

The price of European call option, $C = S\phi(d_+) - Ke^{-r(T-t)}\phi(d_-)$ (2)

The price of European put option, $P = -S\phi(-d_+) + Ke^{-r(T-t)}\phi(-d_-)$ (3)

where,

$$d_+ = \frac{ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t))}{\sigma\sqrt{T-t}},\ d_- = \frac{ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_+ - \sigma\sqrt{T-t}\,,\ \phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$$

1.3. Transformation of Black-Scholes PDE into heat Equation for Approximating Option Price

Substituting $y = \ln\left(\frac{S}{K}\right), \tau = \frac{\sigma^2}{2}(T-t)$ and $u(y,\tau) = \frac{1}{K}e^{-(\alpha y + \beta\tau)}F\left(Ke^y, T - \frac{2\tau}{\sigma^2}\right)$ into equation (1), we get

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2};\quad a \le y \le b\,; 0 \le \tau \le T \tag{4}$$

where S varies from S_{min} to S_{max} and y varies from $a\left(=\ln\left(\frac{S_{min}}{K}\right)\right)$ to $b\left(=\ln\left(\frac{S_{max}}{K}\right)\right)$.

Again substituting $y = a + (b-a)x$ into equation (4), we get

$$\frac{\partial u}{\partial \tau} = \frac{1}{(b-a)^2}\frac{\partial^2 u}{\partial x^2}; \quad 0 \leq x \leq 1\,; 0 \leq \tau \leq \frac{\sigma^2}{2}T \tag{5}$$

$$\text{where, } u(x,\tau) = \frac{1}{K}e^{-[\alpha\{a+(b-a)x\}+\beta\tau]}F\left(Ke^{\{a+(b-a)x\}}, T - \frac{2\tau}{\sigma^2}\right) \tag{6}$$

$\theta = \frac{2r}{\sigma^2}$, $\alpha = \frac{-1}{2}(\theta - 1)$, and $\beta = \frac{-1}{4}(\theta + 1)^2$

1.4. Transformation of Terminal and Boundary Conditions

1.4.1. Transformed Conditions for European Call Options

Initial condition: $u(x,0) = e^{-\alpha\{a+(b-a)x\}}max\left(e^{\{a+(b-a)x\}} - 1,0\right)$ when $0 < x < 1$

Boundary condition: $u(0,\tau) = 0$ and $u(1,\tau) = e^{(1-\alpha)b-\beta\tau} - e^{-\alpha b+\alpha^2\tau}$

1.4.2. Transformed Conditions for European Put Options

Initial condition: $u(x,0) = e^{-\alpha\{a+(b-a)x\}}max\left(1 - e^{\{a+(b-a)x\}}, 0\right)$ when $0 < x < 1$

Boundary condition: $u(0,\tau) = e^{-\alpha a+\alpha^2\tau}$ and $u(1,\tau) = 0$

1.5. Formulation of Du Fort-Frankel Finite Difference Method (DF3DM)

The Du Fort-Frankel finite difference method [11] can be applied to solve different kinds of problems that often occur in finance. This method is a two-step method and require another method for calculating first temporal vector. In this formulation $\frac{\partial u}{\partial \tau}$, $\frac{\partial^2 u}{\partial x^2}$ are approximated by central differencing and u_i^j is replaced by $\frac{u_i^{j+1}+u_i^{j-1}}{2}$.

Thus, discretizing (5) by DF3DM, we get

$$\frac{u_i^{j+1} - u_i^{j-1}}{2\Delta\tau} = \frac{1}{(b-a)^2}\frac{u_{i-1}^j - 2\frac{u_i^{j+1}+u_i^{j-1}}{2} + u_{i+1}^j}{(\Delta x)^2}$$

$$or, \quad u_i^{j+1} = u_i^{j-1} + \frac{2\Delta\tau}{\{\Delta x(b-a)\}^2}\left(u_{i-1}^j - u_i^{j+1} - u_i^{j-1} + u_{i+1}^j\right)$$

$$or,\quad u_i^{j+1} = u_i^{j-1} + 2d\left(u_{i-1}^{j} - u_i^{j+1} - u_i^{j-1} + u_{i+1}^{j}\right),\ \text{where } d = \frac{\Delta\tau}{\{\Delta x(b-a)\}^2}$$

$$\therefore\quad u_i^{j+1} = \frac{2d}{1+2d}\left(u_{i-1}^{j} + u_{i+1}^{j}\right) + \frac{1-2d}{1+2d}u_i^{j-1} \tag{7}$$

where, $i = 0,1,\cdots,n-1\ and\ j = 0,1,\cdots,m-1$

Equation(7) can be written in matrix form as follows

$$\begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ \vdots \\ u_{n-1}^{j+1} \\ u_n^{j+1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2d}{1+2d} & 0 & \cdots & 0 \\ \frac{2d}{1+2d} & 0 & \frac{2d}{1+2d} & 0 & \vdots \\ 0 & \frac{2d}{1+2d} & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \frac{2d}{1+2d} \\ 0 & \cdots & 0 & \frac{2d}{1+2d} & 0 \end{bmatrix} \begin{bmatrix} u_1^{j} \\ u_2^{j} \\ \vdots \\ \vdots \\ u_{n-1}^{j} \\ u_n^{j} \end{bmatrix} + \begin{bmatrix} 2d\,u_0^{j} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 2d\,u_{n+1}^{j} \end{bmatrix} + \begin{bmatrix} (1-2d)u_1^{j-1} \\ (1-2d)u_2^{j-1} \\ \vdots \\ \vdots \\ (1-2d)u_{n-1}^{j-1} \\ (1-2d)u_n^{j-1} \end{bmatrix}$$

1.6. Formulation of Galerkin Weighted Residual Method (GWRM)

The Galerkin weighted residual method [12] is a well-known numerical technique for solving parabolic type PDE. By this method, we can convert a continuous operator problem into a discrete problem.
Let us consider the following parabolic initial boundary value problem:

$$\mu(x)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[\alpha(x)\frac{\partial u}{\partial x}\right] + \beta(x)u(x,t) = f(x,t) \tag{8}$$

Domain: $x_{min} \leq x \leq x_{max}\ ; t \geq t_0$

Boundary Conditions : $u(x_{min},t) = u_{x_{min}}(t),\ u(x_{max},t) = u_{x_{max}}(t);\ t > t_0$

Initial Condition : $u(x,t_0) = u_0(x);\ x_{min} < x < x_{max}$

Suppose a trial solution of (8) is

$$\tilde{u}(x,t) = \sum_{j=1}^{n} a_j(t)\varphi_j(x) \tag{8a}$$

where each of the functions $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$ satisfy the corresponding homogeneous form of the boundary conditions. Now the residual function of (8) is

$$R(x,t) = \mu(x)\frac{\partial \tilde{u}}{\partial t} - \frac{\partial}{\partial x}\left[\alpha(x)\frac{\partial \tilde{u}}{\partial x}\right] + \beta(x)\tilde{u}(x,t) - f(x,t) \tag{8b}$$

So, $R(x,t) = 0$ if and only if $\tilde{u}(x,t)$ coincides with $u(x,t)$ for all x and t in the domain of (8). In this method, we ascertain the parameters $a_1(t),\ a_2(t), \cdots, a_n(t)$ by solving n equations

$$\int_{x_{min}}^{x_{max}} R(x,t)\varphi_i(x)\,dx = 0, \qquad i = 1, 2, \cdots, n \tag{8c}$$

These equations are known as Galerkin equations and the functions $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$ are called basis functions. Applying $(8a)$, $(8b)$ into $(8c)$ we get

$$\sum_{j=1}^{n}\left[\int_{x_{min}}^{x_{max}} \mu(x)\varphi_i(x)\varphi_j(x)\,dx\right]\frac{da_j}{dt} + \sum_{j=1}^{n}\left[\int_{x_{min}}^{x_{max}} \alpha(x)\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx\right]a_j(t)$$
$$+ \sum_{j=1}^{n}\left[\int_{x_{min}}^{x_{max}} \beta(x)\varphi_i(x)\varphi_j(x)\,dx\right]a_j(t)$$
$$= \int_{x_{min}}^{x_{max}} f(x,t)\varphi_i(x)\,dx + \left[\left(\alpha(x)\frac{\partial \tilde{u}}{\partial x}\right)\varphi_i(x)\right]_{x=x_{min}}^{x=x_{max}}$$

which can be written as the following matrix form

$$[C]\left\{\frac{da(t)}{dt}\right\} + [K]\{a(t)\} = [F(t)] \tag{9}$$

where, $C_{ij} = \int_{x_{min}}^{x_{max}} \mu(x)\varphi_i(x)\varphi_j(x)\,dx,$

$$K_{ij} = \int_{x_{min}}^{x_{max}} \alpha(x)\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx + \int_{x_{min}}^{x_{max}} \beta(x)\varphi_i(x)\varphi_j(x)\,dx$$

and $F_i(t) = \int_{x_{min}}^{x_{max}} f(x,t)\varphi_i(x)\,dx + \left[\left(\alpha(x)\frac{\partial \tilde{u}}{\partial x}\right)\varphi_i(x)\right]_{x=x_{min}}^{x=x_{max}}$

In the above notation C_{ij} and K_{ij} are called the heat capacity matrix and stiffness matrix respectively, whereas F_i is known as the load matrix.

Now applying equation (5) and (8) into (1), we get

$$\mu(x) = 1,\ \alpha(x) = \frac{1}{(b-a)^2}, \beta(x) = 0, f(x,\tau) = 0,\ x_{min} = 0\ and\ x_{max} = 1.$$

Thus equation (1) reduces to

$$[C]\left\{\frac{da(\tau)}{d\tau}\right\} + [K]\{a(\tau)\} = 0 \tag{10}$$

where, $C_{ij} = \int_0^1 \varphi_i(x)\varphi_j(x)dx, K_{ij} = \frac{1}{(b-a)^2}\int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx$

1.7. Modified Legendre Polynomials

The Legendre polynomials [13] of order n is,

$$P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x^2)^n], n \geq 1 \tag{11}$$

We make minor changes to the Legendre polynomials in such a way that each of the new polynomials satisfy the analogous homogeneous form of the boundary conditions of equation (5). We call those polynomials as Modified Legendre polynomials, $\tilde{P}_n(x)$ and are defined as follows

$$\tilde{P}_n(x) = x[P_n(x) - 1],\ n \geq 1 \tag{12}$$

Some Modified Legendre polynomials are given below:

$$\tilde{P}_1(x) = x(x - 1), \tilde{P}_2(x) = \frac{3}{2}x(x^2 - 1), \tilde{P}_3(x) = \frac{1}{2}x(5x^3 - 3x - 2),$$

$$\tilde{P}_4(x) = \frac{5}{8}x(7x^4 - 6x^2 - 1), \tilde{P}_5(x) = \frac{1}{8}x(63x^5 - 70x^3 + 15x - 8)$$

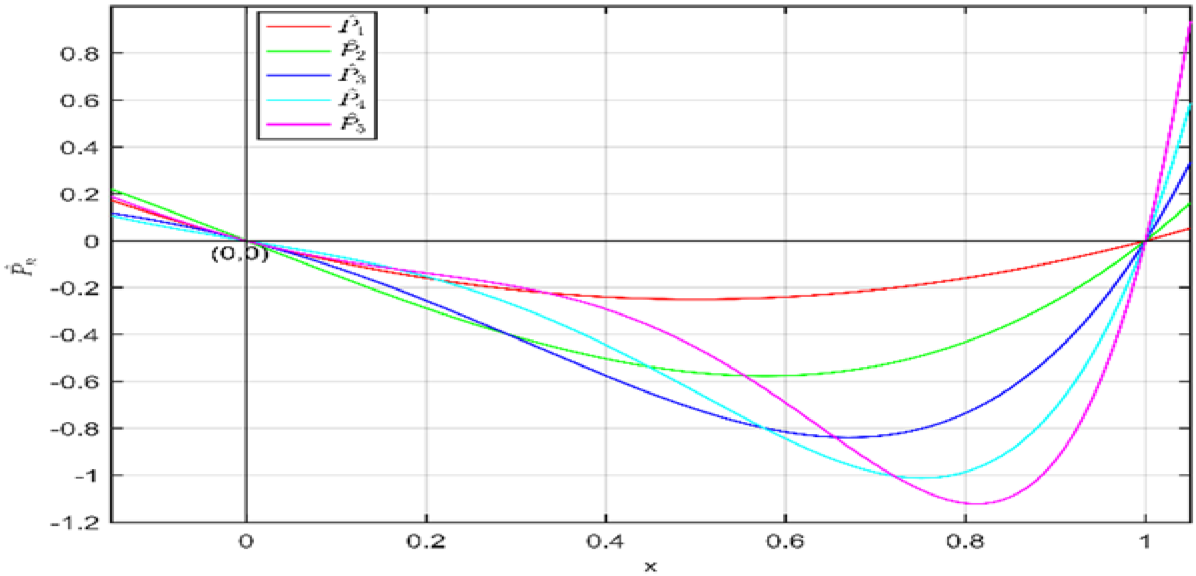


Figure-1. Modified Legendre polynomials $\hat{P}_n(x)$ for $n = 1, 2, 3, 4, 5.$

From Figure 1, we observe that all Modified Legendre polynomials gives 0 at x=0 and 1.

1.8. Numerical Experiments and Result Discussions

In this section, we take two data set for valuing European call option and one data set for finding the value of European put option.

To calculate option values by using DF3DM, we first put $n = m = 250$ (we can also choose $m \neq n$) in equation (7), then using (6) we get approximate values (here we use Crank-Nicolson method to calculate first temporal values). Moreover, approximating option values by using GWRM, we put Modified Legendre polynomials of order $1, 2, 3, and\ 4$ as basis function in equation (10).

Data set 1: For approximating call option value we chose the parameters as follows [6]

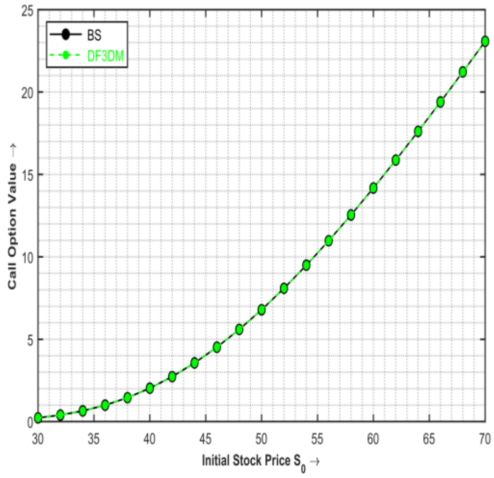
$$K = 50, \quad T = 0.5, \quad r = 0.10, \quad \sigma = 0.4, \quad and\ t = 0.$$

Now substituting the parameters given in data set 1 into equation (2), we get

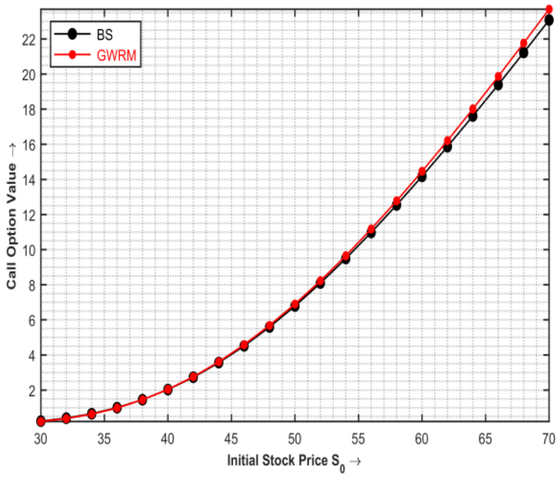
$$C(S_0) = S_0\Phi(d_+) - 47.561471\ \Phi(d\_) \tag{13}$$

where,

$$d_+ = \frac{\ln\left(\frac{S_0}{50}\right)+0.09}{0.282843} \text{ and } d\_ = \frac{\ln\left(\frac{S_0}{50}\right)+0.01}{0.282843}$$



(a)



(b)

Figure-2. Comparing approximate call option value (by using (a) DF3DM, (b) GWRM) with exact (BS) value for data set 1.

Data set 2: For approximating put option value we chose the parameters as follows [4]

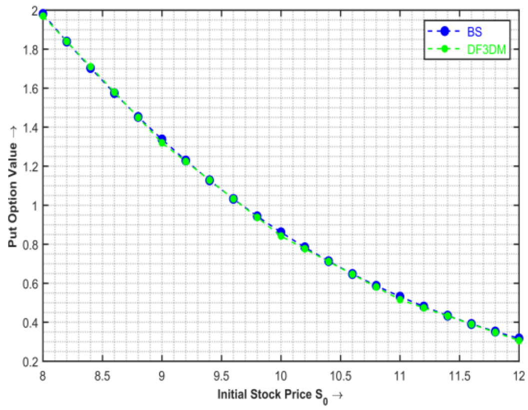
$$K = 10, \quad T = 1/3, \quad r = 0.10, \quad \sigma = 0.45, \quad and\ t = 0.$$

Now substituting the parameters given in data set 2 into equation (3), we get

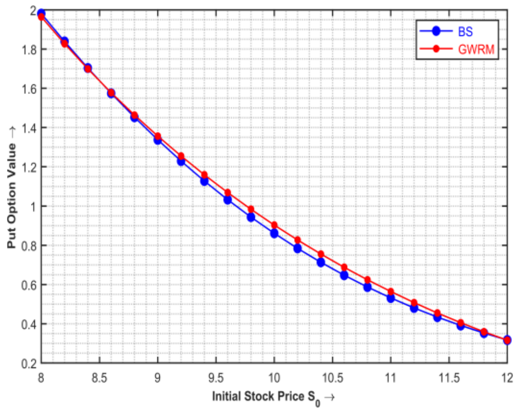
$$P(S_0) = -S_0\phi(-d_+) + 9.672161\phi(-d_-) \tag{14}$$

where

$$d_+ = \frac{\ln\left(\frac{S_0}{10}\right)+0.067083}{0.259808} \text{ and } d_- = \frac{\ln\left(\frac{S_0}{10}\right)-0.000417}{0.259808}$$

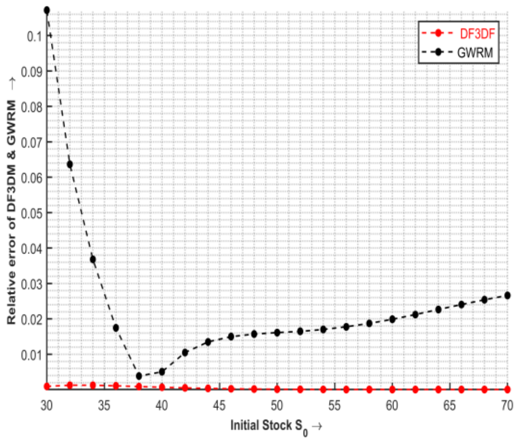


(a)

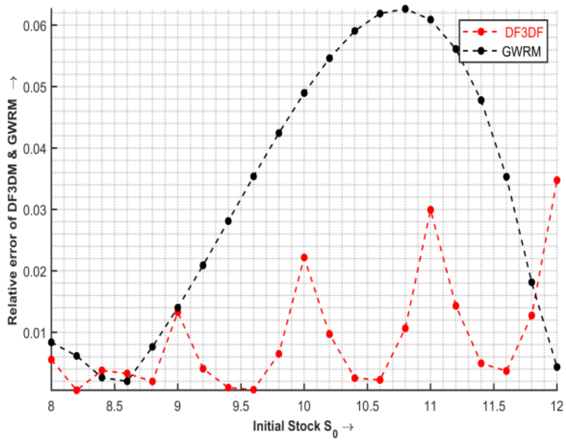


(b)

Figure-3. Comparing approximate put option value (by using (a) DF3DM, (b) GWRM) with exact (BS) value for data set 2.



(a)



(b)

Figure-4. Relative errors of numerical methods (for (a) data set 1, (b) data set 2).

Figures 2 (a) and 3. (a), shows that approximate option prices obtained by DF3DM, nearly equal to BS values for all initial stock prices and from Figures 2. (b) and 3. (b) we observe that when the initial stock price is greater than the strike price then the approximate values obtained by GWRM fluctuates slightly from the BS values.

Figure 4 shows that, relative errors of DF3DM are minimum than relative errors of GWRM almost all the time.

Data set 3: $\left\{S_0 = 65, K = 40,\ r = 0.05,\ \sigma = 0.324366,\ t = 0,\ T = \frac{1}{4},\ \frac{1}{6}\ and\ \frac{1}{12}\right\}$[14]

Now substituting the parameters given in data set 3 into equation (2), we get

$$C(T) = 65 \times \Phi(d_+) - 40 \times e^{-(0.05\times T)}\Phi(d_-)(15)$$

where,

$$d_+ = \frac{0.48551+0.10261\times T}{0.324366\times\sqrt{T}} \text{ and } d_- = \frac{0.48551-0.0026067\times T}{0.324366\times\sqrt{T}}$$

From Table 1, we see that, most of the cases our proposed (present) methods give better results than Reference values.

Table-1. Call option values for data set 3.

T	BS value	Present DF3DM		Present GWRM		Reference [14]	
		Value	Relative error	Value	Relative error	Value	Relative error
$\frac{1}{4}$	25.4993	25.5009	6.33×10^{-05}	25.4969	9.55×10^{-05}	25.4965	1.11×10^{-04}
$\frac{1}{6}$	25.3321	25.3325	1.51×10^{-05}	25.3319	6.08×10^{-06}	25.3319	7.98×10^{-06}
$\frac{1}{12}$	25.1663	25.1663	3.31×10^{-07}	25.1663	2.88×10^{-09}	25.1663	7.95×10^{-07}

2. CONCLUSIONS

From the results of this paper we observe that the results obtained by DF3DM is better than the results obtained by GWRM comparing with the exact (BS) and reference values. Finally, we can conclude that the DF3DM may be a good alternative to solve BS model for European options. Furthermore, we can improve the approximations of these methods by increasing the value of m and n in DF3DM and increasing the number of basis functions in GWRM.

Funding: The authors are grateful to the National Science and Technology (NST), Ministry of Science and Technology, Govt. of People's Republic of Bangladesh, for granting the "NST Fellowship" during the period of research work.
Competing Interests: The authors declare that they have no competing interests.
Acknowledgement: The authors are thankful to the reviewer(s) and editorial board for their comments and suggestions for improvement of the quality of the first version of this manuscript.

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