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ANALYTICAL SOLUTION OF STEADY STATE HEAT CONDUCTION IN A RECTANGULAR PLATE AND COMPARISON WITH THE NUMERICAL FINITE DIFFERENCE METHOD

Obi, Chinwe N.
Department of Mathematics, Federal University of Technology, Owerri, Nigeria.
Email: writechynwe@gmail.com Tel: +2348137745333

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ABSTRACT

The medium of heat conduction is seemingly important in the world today and needs to be well understood. This paper looks at the steady state heat conduction in a rectangular plate characterized by Dirichlet boundary conditions. The steady state heat model is formulated based on some assumptions governing this phenomenon. The model which is an elliptic partial differential equation is solved using both analytic and numerical methods. The separable variable method which is an analytic method gives rise to a closed form solution. A comparative study was made by comparing the accuracy of the Finite Difference Method (FDM) and the separable variable method and the relative errors determined. The results obtained from the separable variable method were close to the numerical method. The FDM gives approximate solution with less time and fitting resilience. It is thus concluded that the FDM method allows control over mensurable error.

Contribution/Originality: The Laplace equation was formulated using a physical phenomenon which is the application of steady heat equation, Certain conditions were prescribed such that the unknown function must satisfy the boundary of the given domain.

1. INTRODUCTION

Most partial differential equations generated from physics govern many qualitatively phys- ical phenomena. Many engineering systems such as steady-state heat conduction exhibit mainly two- or three-dimensional nature. Due to the significance of the steady-state heat equation in a vast variety of fields, there are many approximate solutions of that equation for a vast variety of initial and boundary conditions.

In the last few decades, there has been a lot of literature on solutions of steady state heat equations. The method of fundamental solutions for steady-state heat conduction in nonlinear materials has been investigated by Karageorghis and Lesnic [1]. Dülk and Kovácsházy [2] solved a steady-state heat equation in a two-layer medium problem and then showed that multilayered media can be represented as a hierarchy of two-layered media. Adaptive methods for analytical and numerical solutions of heat diffusion in one-dimension thin rod was investigated by Makhtoumi [3] a comprehensive analysis was carried out between the two methods. Kaushik [4] solved a steady 2-D heat equation with constant thermal conductivity numerically using TDMA technique, a comparison was made between Gauss-Seidel and TDMA. The Finite difference method is based on the differential equation of the heat conduction [5]. The FDM deals with the discretization of space and time such that there is an integer number of points in space and an integer number of times at which we calculate the field variable. In this paper, we formulate a two-dimensional steady state conduction problem in rectangular domain. Finite difference method is adopted to discretize the Laplace equation.

2. MODEL FORMULATION

2.1. Steady-State Temperature Distribution in a Plate

The temperature distribution in a rectangular metal plate is determine based on certain assumption. The plate is covered on its top and bottom surfaces by layers of thermally insulated material such that heat is constrained to flow in two directions, x and y direction. The basic assumptions for the mathematical model include:

(i.) Thermal conductivity is the same at all points of the plate.

(ii.) The plate is sufficiently thin so that we may regulate any heat flow in the distribution.

(iii.) The temperature distribution is steady which means that the temperature at any point in the plate does not vary with time.

Given that the plate whose surface area is dxdy and is subjected to these three assumptions. The heat flows at a rate proportional to the cross-sectional area of the temperature, denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial y}$ and the thermal conductivity k which is assumed at all point, t is the thickness of the plate. The flow of the heat move from high to low temperature. Rate of heat flow into element $x = x_0$ in the $x-$ *direction*,

$$x_0 = -k(tdy)\frac{du}{dx}$$

The gradient at $x_0 + dx$,

$$x_0 + dx = \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\frac{\partial u}{\partial x}dx$$

The rate of heat flow in the element at $(x = x_0 + dx)$

$$x + dx = -k(tdy)[\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\frac{\partial u}{\partial x}dx]$$

The net rate of heat flow into the plate in the x-direction

$$= -k(tdy)\frac{\partial u}{\partial x} + k(tdy)[\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\frac{\partial u}{\partial x}dx]$$

$$= ktdxdy\frac{d^2u}{dx^2}$$

Similarly,

The net rate of heat flow into the element in $y-$ *direction* $=$ $\mathrm{ktdxdy}\frac{\mathrm{d^2u}}{\mathrm{dy^2}}$

The total heat flow into the element's volume is the sum of these net flows in the x and y directions. If heat were generated within the element, this would be added to heat entering by conduction. The sum however must be equal to the heat lost by other mechanism or else, there would be a build up of heat within the element causing its temperature to increase with time. At equilibrium state, the total rate of heat flow into the element including heat generated must be zero, that is,

$$ktdxdy(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}) + Qtdxdy = 0 \tag{1}$$

Where Q is the rate of heat generated per unit volume: Let $Q = 0$,

$$ktdxdy(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}) = 0 \tag{2}$$

For Equation 2 to be valid we argue that,

$$k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right) = 0 \qquad (3)$$

Equation 3 is a Laplace equation.

3. ANALYTIC SOLUTION USING SEPARABLE VARIABLE METHOD

The closed form solution of the elliptic PDE is obtained using separable variable method. The general form of the Laplace equation is expressed as:

$$k \nabla^2 u = 0 \qquad (4)$$

Which is another way of writing Equation 3.

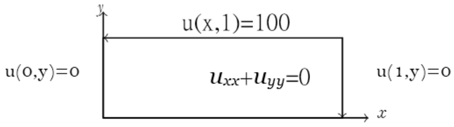
Here k=1 and is the thermal conductivity and u is the temperature. The steady heat is given by:

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \qquad (5)$$

This is a two-dimensional Laplace equation

3.1. Boundary Conditions

The rectangular plate is subjected to Dirichlet boundary conditions at the ends as follows:



u(x,0)=0

$$u(0, y) = 0, \qquad 0 \le x \le 1 \qquad (6)$$

$$u(1, y) = 0 \qquad 0 \le x \le 1 \qquad (7)$$

$$u(x, 0) = 0 \qquad 0 \le y \le 1 \qquad (8)$$

$$u(x, 1) = 100 \qquad 0 \le y \le 1 \qquad (9)$$

Equation 6-10 shows the boundary conditions shows the boundary condition of the rectangular plate

Let,

$$u(x, y) = W_x V_y \qquad (10)$$

Equation 10 is a separable variable is used to solve 5

$$u_{xx} = W''V \qquad (11)$$

$$u_{yy} = V''W \qquad (12)$$

Substituting Equations 11 and 12 into Equation 5, we obtain,

$$W''V + WV'' = 0 \qquad (13)$$

$$\frac{W''}{W} = -\frac{V''}{V} = -\lambda^2 \qquad (14)$$

Where λ is a constant.

From Equation 14, we equate the first and second terms to the term which is a constant as follows:

$$W'' + \lambda^2 W = 0 \qquad (15)$$

$$V'' - \lambda^2 V = 0 \qquad (16)$$

$$W(x) = A\cos\lambda x + B\sin\lambda x \qquad (17)$$

Applying the boundary conditions on Equations 6 and 7 to Equation 17:

$$W_n(x) = B_n \sin n\pi x \qquad (18)$$

Where $\lambda = n\pi$

$$V(y) = C\cosh\lambda y + D\sinh\lambda y \qquad (19)$$

Applying the boundary conditions on Equation 8 to Equation 19:

$$V_n(y) = D_n \sinh n\pi y \qquad (20)$$

Substituting the results in Equations 19 and 20 into Equation 10:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi y \qquad (21)$$

Applying the boundary condition in Equation 9 to Equation 21:

$$100 = u(x,1) = \sum_{n=1}^{\infty} b_n \sin n\pi \sinh n\pi y \qquad (22)$$

$$b_n \sinh n\pi = 2\int_0^1 100 \sin n\pi x\,dx \qquad (23)$$

The final solution is:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi y$$

Where, $b_n = \frac{200}{\sinh n\pi}\int_0^1 \sin n\pi x$

4.THE FINITE DIFFERENCE METHOD

The FDM is a simple technique which can be used to solve the steady-state equation numerically. The solution to the model in Equation 5 with boundary conditions in Equations 6-9 is presented. The aim of the numerical method is to obtain a solution over certain points. The solution domain is divided into number of intervals.

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad (24)$$

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} \qquad (25)$$

Where $h^2 = (\Delta x)^2 = (\Delta y)^2$

The given problem in Equation 5 is discretized by a finite difference method based on a three-point intermediate scheme for the derivative of second order.

$$u_{i-1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1} = 0 \qquad (26)$$

Equation 26 holds at every grid point (x_i, y_j) that is not on the boundary

$i = 1, 2, \cdots, N-1$

$j = 1, 2, \cdots, N-1$

This gives an $(N-1)\times(N-1)$ equations. The Dirichlet boundary conditions are:

$U_{0,j} = 0;$ $\quad U_{N,j} = 0 \quad 0 \le j \le N$

$U_{i,0} = 0;$ $\quad U_{i,N} = g(X_i) = 100 \quad 0 \le i \le N$

The Gauss-sidel iterative scheme for the numerical solution of the given problem is shown in Table 1:

Table 1. Numerical results.

Iterations	$U1,1$	$U2,1$	$U1,2$	$U2,2$
1	0.00	0.00	25.0	25.0
2	6.25	7.813	32.813	35.156
3	10.156	11.328	36.328	36.914
4	12.061	12.244	37.244	37.372
5	12.372	12.436	37.436	37.468
6	12.468	12.484	37.484	37.492
7	12.492	12.495	37.496	37.497
8	12.497	12.499	37.494	37.799

Table 2 presents a comparison between the analytic and approximate method.

Table 2. Comparison between the analytic and approximate method.

X	Y	Analytical	FDM	Relative Error
0.25	0.25	0.013861	0.014782	0.00092
0.50	0.25	0.009674	0.010488	0.00081
0.50	0.50	0.138752	0.147625	0.00887
0.75	0.50	-0.162343	-0.183422	0.021079
0.75	0.75	-0.035623	-0.054276	0.0186533
0.75	1.00	-0.136724	-0.156323	0.019599

Figure 1 depicts the Solution of the steady-state model via FDM.

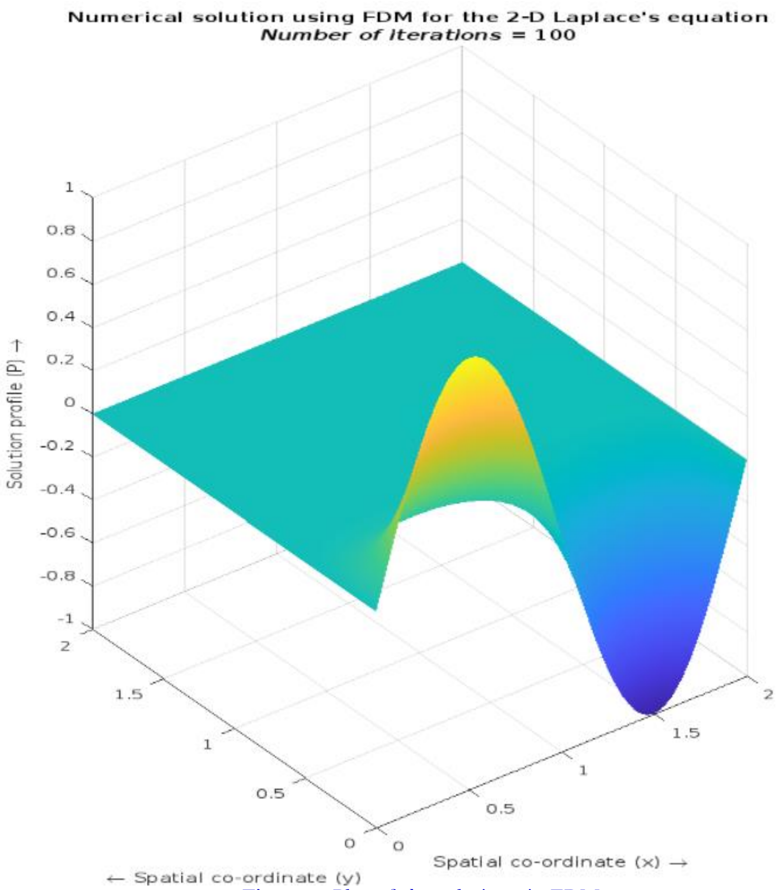


Figure 1. Plot of the solution via FDM.

Figure 2 illustrates the Exact Solution of the steady state -heat model.

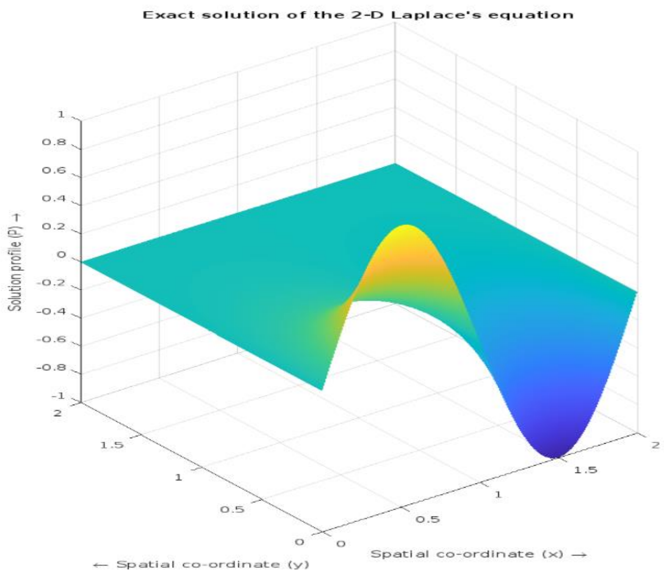


Figure 2. Exact solution plot.

5. RESULT AND DISCUSSION

The system of equations in section 4.0 are used to approximate the steady-state heat model in a rectangular plate. Table 1 lists the approximate results using FDM along- side its exact results which were obtained using separation of variable method. The approximate results and the exact results are present with varying values of x and y in the table in order to obtain the relative error values.

6. CONCLUSION

In this paper, a steady-state heat model was formulated using certain assumption. Two methods, namely, separation of variable and FDM were used to solve the 2-dimensional Laplace equation. The finite difference method is simple to use due to its computational efficiency and simple code structures. The graphs were generated using MATLAB software.

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