International Journal of Mathematical Research
2023 Vol. 12, No. 1, pp. 1-8.
ISSN(e): 2306-2223
ISSN(p): 2311-7427
DOI: 10.18488/24.v12i1.3316

Generalized fractional kinetic equations involving incomplete Aleph - function

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Article History
Received: 6 December 2022
Revised: 31 January 2023
Accepted: 17 February 2023
Published: 9 March 2023

Keywords
Fractional kinetic equations
Incomplete Aleph- function
Laplace transform
Riemann-liouville fractional integral operator
Sumudu transform.

ABSTRACT

Due to the great importance of the fractional kinetic equations, many authors discussed the generalizations of fractional kinetic equation involving various special functions. The purpose of this paper is to obtain the new generalization of fractional kinetic equation pertaining to the incomplete Aleph-function. The solution of the fractional kinetic equations obtained here by using Laplace and Sumudu transforms method. The Riemann-Liouville fractional integral operator is used to obtain the required results. The Solution of the generalized fractional kinetic equation are obtained by using the definition of incomplete Aleph function. The result discussed here can be used for the study of the chemical composition change in stars like the Sun. The solution rendered here are in compact forms suitable for numerical computation. Some special cases involving incomplete I –functions and incomplete H –functions are also considered.

Contribution/Originality: In this paper author obtain the solution of fractional kinetic equation using incomplete Aleph function which is new and more significant in discussing some astrophysical problems. In order to obtain the solution of the fractional kinetic equations we have used the lemmas involving the Laplace and Sumudu Transforms and their inverse transforms.

1. INTRODUCTION

The fundamental equations of mathematical physics and natural science, the kinetic equations define the continuity of matter motion. Numerous significant problems in physics and astrophysics have been described and solved using fractional kinetic equations incorporating a large range of special functions. Numerous fractional operators were used in the extension and generalization of fractional kinetic equations [1-4]. The author presents a more generalized form of the fractional kinetic equation regarding incomplete Aleph-function in light of the effectiveness and significant role of the kinetic equation in some astrophysical situations.

Haubold and Mathai [1] created the fractional differential equation between the rate of change of the reaction, the destruction rate, and the production rate, which is stated as follows:

$$\frac{dN}{dt} = -d(N_t) + p\,(N_t)\ , \tag{1}$$

In equation N=N(t) the rate of reaction, d = d(N) the rate of destruction, p = p(N) the rate of production and N_t denotes the function defined by $N_t\,(N^*) = N(t - t^*)\,,\ t^* > 0\,.$

Haubold and Mathai [1] studied a special case of this equation, when instance fluctuation in quantity N(t) are neglected, is given by the equation:

$$\frac{dN_i}{dt} = -c_i N_i(t), \qquad (2)$$

With $N_t(t = o) = N_0$ is the number density of species i at time t=0; constant $c_i > 0$, known as standard kinetic equation. The solution of the above standard kinetic Equation 2 is given by:

$$N_i(t) = N_0 e^{-c_i t} \qquad (3)$$

An alternative form the same equation can be obtained on integration:

$$N(t) - N_0 = c \; {}_0D_t^{-1} N(t), \qquad (4)$$

Where ${}_0D_t^{-1}$ is the standard integral operator. Haubold and Mathai [1] have given the fractional generalization of the standard kinetic Equation 4 as:

$$N(t) - N_0 = c^v \; {}_0D_t^{-v} N(t), \qquad (5)$$

Where ${}_0D_t^{-v}$ is the well known Riemann-Liouville fractional integral operator Oldhman and Spanier [5]; Samko, et al. [6]; Miller and Ross [7]; Srivastava and Saxena [8]) defined by :

$${}_0D_t^{-v} = \frac{1}{\Gamma(v)} \int_0^t (t-y)^{v-1} f(y) dy, \qquad Re(v) > 0 \qquad (6)$$

The solution of the fractional kinetic Equation 5 is given by Haubold and Mathai [1] as:

$$N(t) = N_0 \sum_{K=0}^{\infty} \frac{(-1)^K}{\Gamma(vK+1)} (ct)^{vK} \quad .$$

Sumudu transform defined by Watugala [9] for functions of exponential order over the set of functions,

$$\mathcal{A} = \left\{ f(t) : \exists\, M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, if\; t \in (-1)^j [0, \infty) \right\}. \qquad (7)$$

The Sumudu transform is defined by:

$$G(h) = S[f(t)] = \int_0^{\infty} f(ht)\, e^{-t}\, dt \;, \quad h \in (-\tau_1, \tau_2). \qquad (8)$$

The Riemann-Liouville fractional integral of order v is defined by Miller and Ross [7]:

$${}_0D_t^{-v} N(x,t) = \frac{1}{\Gamma(v)} \int_0^t (t-y)^{v-1} N(x,y) dy, \qquad Re(v) > 0 \qquad (9)$$

The Sumudu transform of the Riemann-Liouville fractional integral of order v is defined as Belgacem, et al. [10]; Saxena, et al. [2]:

$$S\{ {}_0D_t^{-v} f(t); h\} = h^v \bar{f}(h) \qquad (10)$$

The Aleph function was first established in the 19th century by Sudland, et al. [11], and numerous fascinating findings and prospective applications in science, technology, and other disciplines were afterwards debated and extended by various authors.

A recent investigation and comprehensive research of the incomplete Pochhammer symbols was conducted by Srivastava, et al. [12]. They also discussed derivative formulae, Mellin-Barnes contour integral, generalized incomplete hypergeometric functions, and their integral representation. They also discussed potential applications in the areas of communication theory, groundwater pumping models, and probability theory.

The well-known incomplete gamma functions [13] are defined as:

$$\gamma(v,y) = \int_0^y u^{v-1} e^{-u}\, du\,, \;\; \{ Re(v) > 0;\; y \geq 0 \qquad (11)$$

$$\Gamma(v,y) = \int_y^{\infty} u^{v-1} e^{-u}\, du\,, \; \{ \; y \geq 0\,;\; Re(v) > 0 \; when \;\; y = 0 \, \}, \qquad (12)$$

Respectively, satisfy the subsequent decomposition formula:

$$\gamma(v,y) + \Gamma(v,y) = \Gamma(v) \;\; \{\mathrm{Re(v)} > 0\}$$

Recently Kumar Bansal, et al. [14] introduce and investigate the incomplete Aleph –functions:

${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$with the help of the incomplete gamma functions as follows:

$$ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z) = {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z\left|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right) $$

$$ = \frac{1}{2\pi i}\int_L K(s,y)\ z^{-s}\ \mathrm{ds}, \qquad (13) $$

Where $z \neq 0, and$

$$ K(s,y) = \frac{\Gamma(1-a_1-\alpha_1 s\ ,y)\ \prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\ \prod_{j=2}^{n}\Gamma(1-a_j-\alpha_j s)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\right]} \qquad (14) $$

and

$$ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z) = {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z\left|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right) $$

$$ = \frac{1}{2\pi i}\int_L L(s,y)\ z^{-s}\ \mathrm{ds}, \qquad (15) $$

Where $z \neq 0, and$

$$ L(s,y) = \frac{\gamma(1-a_1-\alpha_1 s\ ,y)\ \prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\ \prod_{j=2}^{n}\Gamma(1-a_j-\alpha_j s)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\right]} \qquad (16) $$

The incomplete $\aleph$ -functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ in (13) and (15) exists for all $y \geq 0$ under the set of conditions given below:

The contour L in the complex s-plane extends from $\gamma - i\infty$ **to** $\gamma + i\infty$, $\gamma\epsilon\mathbf{R}$ and the poles of the gamma functions $\Gamma(1-a_j-\alpha_j s)$, $j = \overline{1,n}$ do not exactly match with the poles of the gamma function $\Gamma(a_j+\alpha_j s), j = \overline{1,m}$. The parameters p_i , q_i are non negative integers satisfying $0 \leq n \leq p_i$, $0 \leq n \leq q_i$ for $i = \overline{1,r}$. The parameters a_j, α_j , a_{ji} , α_{ji} are positive numbers and b_j, β_j , b_{ji} , β_{ji} are complex. All poles of $K(s,y)$ and $L(s,y)$ are supposed to be simple and the empty product is treated as unity.

$$ \Psi_i > 0\,, |\arg(z)| < \frac{\pi}{2}\Psi_i \quad , i = \overline{1,r} $$

$$ \Psi_i \geq 0\,, |\arg(z)| < \frac{\pi}{2}\Psi_i \quad and\ \ Re(\Phi_i) + 1 < 0\ , $$

$$ \text{where} \quad \Psi_i = \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \tau_i\left(\sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=m+1}^{q_i}\beta_{ji}\right), $$

$$ \Phi_i = \sum_{j=1}^{m}b_j - \sum_{j=1}^{n}a_j + \tau_i\left(\sum_{j=m+1}^{q_i}\alpha_{ji} - \sum_{j=n+1}^{p_i}\beta_{ji}\right) + \frac{1}{2}(p_i - q_i) \qquad , i = \overline{1,r} $$

2. INTEGRAL TRANSFORMS OF ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$

To solve generalized fractional kinetic equation, we use the Laplace transforms and Sumudu Transforms of incomplete Aleph –functions which are as follows:

Lemma 2.1. If

$$ \Psi_i > 0\,,\ \ \rho > 0, \qquad |\arg(z)| < \frac{\pi}{2}\Psi_i \ , \qquad Re(\Phi_i) + 1 < 0\,,\ \ i = \overline{1,r} $$

$$ Re\left(\sigma + \rho \min_{1\leq j\leq m}\left\{\frac{Re(b_j)}{\beta_j}\right\}\right) > 0\,, Re(h) > 0\,, c > 0\ and\ \ y \geq 0\ , $$

Then the Laplace transform of the incomplete $\aleph$ -functions Choi and Kumar [4] is given by:

$$\mathcal{L}\left\{ t^{\sigma-1}\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right); h \right\}$$

$$= h^{-\sigma}\ {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left(ch^{-\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right) \tag{17}$$

and

$$\mathcal{L}\left\{ t^{\sigma-1}\ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right); h \right\}$$

$$= h^{-\sigma}\ {}^{(\gamma)}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left(ch^{-\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right), \tag{18}$$

Provided that each member in (17) and (18) exist.

Lemma 2.2. From the above lemma, it is obvious that

$$\mathcal{L}^{-1}\left\{ h^{-\sigma}\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ch^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right); t \right\}$$

$$= t^{\sigma-1}\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i+1,\tau_i;r}\left(ct^{-\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (1-\sigma,\rho),(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right) \tag{19}$$

and

$$\mathcal{L}^{-1}\left\{ h^{-\sigma}\ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ch^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right); t \right\}$$

$$= t^{\sigma-1}\ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i+1,\tau_i;r}\left(ct^{-\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (1-\sigma,\rho),(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right) \tag{20}$$

Lemma 2.3. If

$$\Psi_i > 0,\ \ \rho > 0, \qquad |\arg(z)| < \frac{\pi}{2}\Psi_i \ , \qquad Re(\Phi_i) + 1 < 0,\ \ i = \overline{1,r}$$

$$Re\left(\sigma + \rho \min_{1\le j\le m}\left\{\frac{Re(b_j)}{\beta_j}\right\}\right) > 0,\ h \in (-\tau_1,\tau_2),\ Re(h) > 0, c > 0 \text{ and } y \ge 0\ ,$$

Then the Sumudu transform of the incomplete $\aleph$ -functions Choi and Kumar [4] is given by:

$$\mathcal{S}\left\{ t^{\sigma-1}\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right); h \right\}$$

$$= h^{\sigma-1}\ {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left(ch^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right) \tag{21}$$

and

$$\mathcal{S}\left\{ t^{\sigma-1}\ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right); h \right\}$$

$$= h^{\sigma-1}\ {}^{(\gamma)}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left(ch^{-\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right), \tag{22}$$

Provided that each members in (21) and (22) exist.

Lemma 2.4. From the above lemma, it is obvious that:

$$\mathcal{S}^{-1}\left\{ h^{\sigma}\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ch^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right); t \right\}$$

$$= t^{\sigma-1}\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i+1,\tau_i;r}\left(ct^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (1-\sigma,\rho),(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right) \tag{23}$$

and

$$\mathcal{S}^{-1}\left\{ h^{\sigma}\ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ch^{\rho}\left| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right); t \right\}$$

$$= t^{\sigma-1}\ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i+1,\tau_i;r}\left(ct^{\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(1-\sigma,\rho),(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right) \tag{24}$$

Where $Re\left(\sigma+\rho\min\limits_{1\le j\le m}\left\{\frac{1-Re(b_j)}{\beta_j}\right\}\right)>0$

3. SOLUTION OF GENERALIZED FRACTIONAL KINETIC EQUATION

Theorem 3.1.

If $\sigma,\rho,v,l>0$, $Re(h)>0$, with $|h|<l^{-1}$ and $y\ge 0$, $c\ne l$, $\tau_i>0$, $i=1,\ldots,r$

then the solution of the generalized fractional kinetic equation:

$$N(t)-N_0\,t^{\sigma-1}\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)=-l^{v}\,{}_0D_t^{-v}N(t), \tag{25}$$

is solvable and its solution is given by:

$$N(t)=$$

$$N_0\,t^{\sigma-1}\sum_{k=0}^{\infty}(-1)^k(lt)^{kv}\ {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left(ct^{\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(1-\sigma-vk,\rho),[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right) \tag{26}$$

Proof. Taking Laplace transform on both sides of (25) and using (17), we get

$$\mathcal{L}\{N(t)\}-N_0\mathcal{L}\left\{t^{\sigma-1}\ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right\}$$

$$=-l^{v}\mathcal{L}\{{}_0D_t^{-v}N(t)\},$$

$$\bar{N}(h)-N_0\,h^{-\sigma}\ {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left(ch^{-\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)$$

$$=-l^{v}\ h^{-v}\ \bar{N}(h)$$

Thus we have

$$\bar{N}(h)=N_0\frac{h^{-\sigma}}{(1+l^{v}\ h^{-v})}\ {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left(ch^{-\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right) \tag{27}$$

Now taking inverse Laplace transform on both sides and using lemma (2.2), we have

$$N(t)=N_0\,t^{\sigma-1}\sum_{k=0}^{\infty}(-1)^k(lt)^{kv}\ {}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left(ct^{\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(1-\sigma-vk,\rho),[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)$$

This completes the proof of Theorem (3.1).

Theorem 3.2.

If $\sigma,\rho,v,l>0$, $Re(h)>0$, with $|h|<l^{-1}$ and $y\ge 0$, $c\ne l$, $\tau_i>0$, $i=1,\ldots,r$

Then the solution of the generalized fractional kinetic equation

$$N(t)-N_0\,t^{\sigma-1}\ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right)=-l^{v}\,{}_0D_t^{-v}N(t), \tag{28}$$

Is solvable and its solution is given by:

$$N(t)=$$

$$N_0\,t^{\sigma-1}\sum_{k=0}^{\infty}(-1)^k(lt)^{kv}\ {}^{(\gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left(ct^{\rho}\left|\begin{matrix}(a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(1-\sigma-vk,\rho),[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right) \tag{29}$$

Proof. Proof is similar to the proof of theorem (3.1).

4. ALTERNATE SOLUTION OF GENERALIZED FRACTIONAL KINETIC EQUATION

In this part, we solve the generalized fractional kinetic equation by using Sumudu transform.

Theorem 4.1.

If $\boldsymbol{\sigma, \rho, v, l > 0\,, Re(h) > 0, with\ \ |h| < l^{-1}\ \ and\ \ y \geq 0\ , c \neq l\,, \tau_i > 0\,, i = 1, \dots, r}$

then the solution of the generalized fractional kinetic equation

$$\boldsymbol{N(t) - N_0\, t^{\sigma-1}\, {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\middle|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right) = -l^{v}\,{}_0D_t^{-v}\,N(t),} \tag{30}$$

Is solvable and its solution is given by:

$$N(t) = N_0\, t^{\sigma-1}\sum_{k=0}^{\infty}(-1)^k\,(lt)^{kv}\,{}^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left(ct^{\rho}\middle|\begin{matrix}(a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(1-\sigma-vk,\rho),[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right) \tag{31}$$

Theorem 4.2.

If $\boldsymbol{\sigma, \rho, v, l > 0\,, Re(h) > 0, with\ \ |h| < l^{-1}\ \ and\ \ y \geq 0\ , c \neq l\,, \tau_i > 0\,, i = 1, \dots, r}$

then the solution of the generalized fractional kinetic equation:

$$\boldsymbol{N(t) - N_0\, t^{\sigma-1}\, {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\middle|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right) = -l^{v}\,{}_0D_t^{-v}\,N(t),} \tag{32}$$

is solvable and its solution is given by:

$$N(t) = N_0\, t^{\sigma-1}\sum_{k=0}^{\infty}(-1)^k\,(lt)^{kv}\,{}^{(\gamma)}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left(ct^{\rho}\middle|\begin{matrix}(a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(1-\sigma-vk,\rho),[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right) \tag{33}$$

Proof. Proofs of the theorems (4.1) and (4.2) are similar to proofs of theorems (3.1) .

5. SPECIAL CASES

On setting $\boldsymbol{y = 0}$,the incomplete Aleph –function reduces to Aleph –function introduced by Südland and we obtain the results derived by Choi and Kumar [4].

If we take $\boldsymbol{\tau_i} = 1$,then (13) and (15) reduce to the incomplete *I*- functions introduced by Kumar Bansal, et al. [14] and we obtain the following results:

Corollary 5.1.

If $\boldsymbol{\sigma, \rho, v, l > 0\,, Re(h) > 0, with\ \ |h| < l^{-1}\ \ and\ \ y \geq 0\ , c \neq l\,, i = 1, \dots, r}$

then the solution of the equation

$$\boldsymbol{N(t) - N_0\, t^{\sigma-1}\, {}^{(\Gamma)}I^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\middle|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right) = -l^{v}\,{}_0D_t^{-v}\,N(t),} \tag{34}$$

Is given by:

$N(t) =$

$$N_0\, t^{\sigma-1}\sum_{k=0}^{\infty}(-1)^k\,(lt)^{kv}\,{}^{(\Gamma)}I^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left(ct^{\rho}\middle|\begin{matrix}(a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(1-\sigma-vk,\rho),[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right) \tag{35}$$

Corollary 5.2

If $\sigma, \rho, v, l > 0\,, Re(h) > 0, with\ \ |h| < l^{-1}\ \ and\ \ y \geq 0\ , c \neq l\,, i = 1, \dots, r$

then the solution of the equation

$$\boldsymbol{N(t) - N_0\, t^{\sigma-1}\, {}^{(\gamma)}I^{m,n}_{p_i,q_i,\tau_i;r}\left(ct^{\rho}\middle|\begin{matrix}(a_1,\alpha_1,y),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right) = -l^{v}\,{}_0D_t^{-v}\,N(t),} \tag{36}$$

is given by:

$$N(t) = \; N_0\, t^{\sigma-1} \sum_{k=0}^{\infty} (-1)^k\, (lt)^{kv}\; {}^{(\gamma)}I^{m,n+1}_{p_i+1,q_i+1,\tau_i;r} \left(ct^{\rho} \middle| \begin{matrix} (a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,n},[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(1-\sigma-vk,\rho),[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right) \quad (37)$$

If we set $\boldsymbol{\tau_i}$= 1 and r = 1, then (13) and (15) reduce to the incomplete *H*- functions introduced by Srivastava, et al. [15] and we obtain the following results:

Corollary 5.3.

If $\boldsymbol{\sigma,\rho,v,l > 0\,, Re(h) > 0, with\quad |h| < l^{-1}\quad and\;\; y \geq 0}$,$\boldsymbol{c \neq l}$,

then the solution of the equation

$$\boldsymbol{N(t) -\; N_0\, t^{\sigma-1}\, \Gamma^{m,n}_{p,q} \left(ct^{\rho} \middle| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,p} \\ (b_j,\beta_j)_{1,m} \end{matrix} \right) = -l^{v}\, {}_0D_t^{-v}\, N(t),} \quad (38)$$

Given by:

$$N(t) = \; N_0\, t^{\sigma-1} \sum_{k=0}^{\infty} (-1)^k\, (lt)^{kv}\, \Gamma^{\,m,n+1}_{\,p+1,q} \left(ct^{\rho} \middle| \begin{matrix} (a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,p} \\ (1-\sigma-vk,\rho),(b_j,\beta_j)_{1,m} \end{matrix} \right) \quad (39)$$

Corollary 5.4.

If $\sigma,\rho,v,l > 0\,, Re(h) > 0, with\quad |h| < l^{-1}\quad and\;\; y \geq 0$,$c \neq l$,

then the solution of the equation

$$\boldsymbol{N(t) -\; N_0\, t^{\sigma-1}\, \gamma^{m,n}_{p,q} \left(ct^{\rho} \middle| \begin{matrix} (a_1,\alpha_1,y),(a_j,\alpha_j)_{2,p} \\ (b_j,\beta_j)_{1,m} \end{matrix} \right) = -l^{v}\, {}_0D_t^{-v}\, N(t),} \quad (40)$$

Given by:

$$N(t) = \; N_0\, t^{\sigma-1} \sum_{k=0}^{\infty} (-1)^k\, (lt)^{kv}\, \gamma^{\,m,n+1}_{\,p+1,q} \left(ct^{\rho} \middle| \begin{matrix} (a_1,\alpha_1,y),(1-\sigma,\rho),(a_j,\alpha_j)_{2,p} \\ (1-\sigma-vk,\rho),(b_j,\beta_j)_{1,m} \end{matrix} \right) \quad (41)$$

6. CONCLUSION

In this article, we have explored a new generalization of fractional kinetic equation. The solutions of fractional kinetic equation have been developed in terms of incomplete Aleph –function. Also, the particular cases hold the incomplete H-function, the incomplete *I*-function and familiar Aleph-function were discussed. The new generalized fractional kinetic equations presented here are general in nature and can be used in Statistical Mechanics and other branches of Mathematics.

Funding: This study received no specific financial support.
Competing Interests: The author declares that there are no conflicts of interests regarding the publication of this paper.

REFERENCES

[1] H. J. Haubold and A. M. Mathai, "The fractional kinetic equation and thermonuclear functions," *Astrophysics and Space Science*, vol. 273, no. 1, pp. 53-63, 2000.

[2] R. Saxena, A. Mathai, and H. Haubold, "On fractional kinetic equations," *Astrophysics and Space Science*, vol. 282, no. 1, pp. 281-287, 2002.

[3] V. Chaurasia and J. Singh, "Application of Sumudu transform in fractional kinetic equations, Gen," *Mathematical Notes*, vol. 2, no. 2, pp. 86-95, 2011.

[4] J. Choi and D. Kumar, "Solutions of generalized fractional kinetic equations involving Aleph functions," *Mathematical Communications*, vol. 20, no. 1, pp. 113-123, 2015.

[5] K. B. Oldhman and J. Spanier, *The fractional calculus: Theory and applications of differentiation and integration to arbitrary order*. New York: Academic Press, 1974.

[6] G. Samko, K. A. A., and O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*. New York: Gordon and Breach Science Publishers, 1993.

[7] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*. New York: John Wiley and Sons, 1993.

[8] H. M. Srivastava and R. K. Saxena, "Operators of fractional integration and their applications," *Applied Mathematics and Computation*, vol. 118, no. 1, pp. 1-52, 2001. https://doi.org/10.1016/s0096-3003(99)00208-8

[9] G. K. Watugala, "Sumudu transform- a new integral transform to solve differential equations and control engineering problems," *Mathematics Engineering Industrial*, vol. 6, no. 4, pp. 319-329, 1998.

[10] F. B. M. Belgacem, A. A. Karaballi, and S. L. Kalla, "Analytical investigations of the Sumudu transform and applications to integral production equations," *Mathematical Problems in Engineering*, vol. 2003, no. 3, pp. 103-118, 2003. https://doi.org/10.1155/s1024123x03207018

[11] N. Sudland, B. Baumann, and T. F. Nonnenmacher, "Open problem: Who knows about the aleph (ℵ)-functions?," *Fractional Calculus and Applied Analysis*, vol. 1, pp. 401-402, 1988.

[12] H. Srivastava, M. A. Chaudhry, and R. P. Agarwal, "The incomplete Pochhammer symbols and their applications to hypergeometric and related functions," *Integral Transforms and Special Functions*, vol. 23, no. 9, pp. 659-683, 2012. https://doi.org/10.1080/10652469.2011.623350

[13] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST digital library of mathematical functions; U.S. Department of commerce, national institute of standards and technology: Washington, DC, USA*. Cambridge, UK: Cambridge University Press, 2010.

[14] M. Kumar Bansal, D. Kumar, K. Nisar, and J. Singh, "Certain fractional calculus and integral transform results of incomplete ℵ-functions with applications," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 8, pp. 5602-5614, 2020. https://doi.org/10.1002/mma.6299

[15] H. Srivastava, R. Saxena, and R. Parmar, "Some families of the incomplete H-functions and the incomplete $$\overline H $$ H¯-functions and associated integral transforms and operators of fractional calculus with applications," *Russian Journal of Mathematical Physics*, vol. 25, no. 1, pp. 116-138, 2018. https://doi.org/10.1134/s1061920818010119