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The process of mathematics students' reflective abstraction in solving continuous random variable problems in the probability course

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ABSTRACT

Reflective abstraction is a mental mechanism in mental constructs in which all the structures of mathematical logic are developed in the mind of the individual. The mental mechanisms in question are interiorization, coordination, reversal, encapsulation, and generalization. This research is classified as exploratory research with a qualitative descriptive approach. This research data was taken from three fifth-semester mathematics students. The instruments used in this study were researchers as key instruments, essay tests, and interview guidelines. Data analysis included: 1) preparing the data for analysis, 2) reading the entire data, 3) encoding the data, 4) describing the data and presenting it in the narrative, and 5) interpreting the data and testing the validity of the data using data triangulation. The research found that (1) The first participant went through four components of reflective abstraction: interiorization – coordination – encapsulation – generalization. (2) The second participant went through five components of reflective abstraction thinking namely interiorization – coordination – reversal – encapsulation – generalization. (3) The third participant failed to solve the problem because the thought process in his mental mechanism had not yet manifested coordination, encapsulation, and generalization. Researchers found that the mental construction that the first participant had gone through could illustrate that the minimum mental construction that must be passed in solving the problem of continuous random variables required components-interiorization, coordination, encapsulation, and generalization.

Contribution/Originality: The study contributes how students' reflective abstraction processes help in solving problems related to continuous random variables. This study showed how reflective abstraction can be used as a mental mechanism in mental constructs where all structures of mathematical logic can be developed in the mind of the individual.

1. INTRODUCTION

Probability and statistics play an important role in the mathematics curriculum because it is studied at every level of education, from basic education to higher education. Several previous studies have shown that probability and statistics are widely applied in various fields of science and technology, such as health, insurance, politics, security, medicine, etc. (Chi, 2022; Elbehary, 2021; Junior, Neto, & Saito, 2022; Malovichko, 2021). Humans in the 21st century must understand something about probability and statistics (Milinkovi & Radovanovic, 2021). In everyday life, a person is often faced with some facts involving probabilities; for example, in coin tossing, rolling dice, games, weather forecasts, sports strategies, insurance options, recreational activities, making a business and so on. Probability can be described as the basis or foundation of statistical theory and its application (Lee, Doerr, Tran, & Lovett, 2016). Probability is a tool for describing, measuring, and modeling randomness to help people make sensible decisions under random conditions (Gong & He, 2017). Any misunderstandings about probability can affect a person's decision-making in important situations (Sharna, Sharma, Doyle, Marcelo, & Kumar, 2021). Such is the importance of probability that students of mathematics as prospective teachers need adequate abilities to transfer back the knowledge they have to students. It is important to be equipped with how to teach mathematics and deal with students who have different abilities and levels. The learning process begins with students calling available activities to complete an assignment. Through the sequence of assignments, students come to be able to anticipate the results of this activity (that is, without having to do activities) (Kara, Simon, & Placa, 2018). Prayekti, Nusantara, Sudirman, and Rofiki (2020) state that the mental model is an internal representation that involves recalling and processing of the information that memory has aimed at solving problems. The mental model of the student has a huge role in problem-solving skills. Students face mathematical problem situations by building and reconstructing mental structures (Cetin & Dubinsky, 2017). It can be said that the process of reflective abstraction in solving mathematical problems is an important capital that students need to have.

As an effort to support the teaching and learning process, an understanding of how mathematical conceptions are constructed is required. It is very important to have an accurate description of the processes used to develop mathematical knowledge, especially in the course of probability theory. The mental structure that occurs when understanding the concept of probability is reviewed from the Action-Process-Object-Shema (APOS) theory (Syamsuri & Santosa, 2021). APOS theory is based on Piaget (2014) theory regarding reflective abstraction, or the process of building mental notions of mathematical knowledge by individuals during cognitive development. The process of reflective abstraction that was observed in this study helped to understand the mental construction of students of mathematics while solving problems regarding continuous random variables. The five mental mechanisms that also led to the mental structure of APOS were interiorization, coordination, encapsulation, generalization, and reversal (Fuady, Purwanto, & Rahardjo, 2020; Nisa, Waluya, & Mariani, 2021).

Interiorization is the process of understanding the information and instructions involved, for example carefully reading information about questions and instructions. Coordination connects several separate operations to form new mental processes. Encapsulation: the process of considering, understanding, and summarizing concepts e.g. from the concept of shared probability, which is a mental object as a dynamic or static process. The generalization is that once an individual summarizes an idea, it is applied to a broader set of mathematical problems. Reversal is the process of returning to an object to agree on whether a given problem can be solved.

It was premised that a reflective abstraction would be more likely to occur when students worked on problem-solving questions than practice questions (Nisa, Waluya, Kartono, & Mariani, 2021). Reflective abstraction signaled a more complex procedure. While the individual receives only info with empirical abstraction, the student began to apply actions to objects with reflective abstraction (Özmen & Mumcu, 2020). Piaget (2014) describes reflective abstraction as coordination of actions with each action as a part of an already available concept (Kara et al., 2018). However, Piaget's belief that reflective abstraction involved the reorganization of actions or operations at a higher level, was challenged as something that did not always happen in the learning of mathematics (Norton, 2018).

Previous research has discussed the process of reflective abstraction in solving mathematical problems, especially at the university level. Sopamena (2018) reported on students' thought processes in solving mathematical problems (limit concepts) based on reflective abstractions. Irawati, Hidayah, and Muksar (2021) described students' abstraction thinking ability in the algebraic structure course. Nisa, Waluya, and Mariani (2020) explored the implementation of APOS in students to bring out reflective abstractions in the Riemann Sum material. Aminah, Sukestiyarno, Wardono, and Cahyono (2022) found that while searching solutions to diophantine linear equation problems, students use the reflective abstraction thinking component.

From several studies that have discussed reflective abstraction thinking, the concept of limits, functions, linear equations and like have received much attention. There is little attention given to the process of abstraction of reflection in solving problems and in other advanced courses such as probability theory and mathematical statistics. Piaget and Inhelder (2013) which focuses on the developmental stage, also reveals insights into students' thinking about uncertainty, data, sample space, randomness, and possibilities (Grinstein & Lipsey, 2001). Research related to probability education, especially in prospective teacher students, is still limited and needs to be improved (Dayal & Sharma, 2020; Kuznetsova, 2019). Based on this description, reflective abstraction is a suitable framework for testing learning in mathematics, especially in advanced mathematical learning, such as probability theory.

Probability materials studied at the college level include basic concepts of probability spaces, mutually free events, discrete and continuous random variable, and their distributions, the value of expectation and variation of random variables, the joint distribution of random variables, and several other studies (Astuti, Anggraeni, & Setyawan, 2020). The achievement of this probability theory course is expected that prospective teacher students can transfer the knowledge gained to students, then apply it in everyday life.

The gap that occurs especially in research places is that there are still many students who consider probability to be a difficult course. Several studies identify probability, randomness, and uncertainty as some of the most elusive mathematical concepts (Ingram, 2022; Milinkovi & Radovanovic, 2021). Students have difficulty understanding the problem and remembering the formulas in solving the problem. Probability is a complex course, which has systematic and logical characteristics (Wang & Xu, 2018). In addition, the inability to learn certain concepts causes students to not succeed in making appropriate mental constructs, such as the concepts of random samples and statistical inference (Borovcnik, 2017). Another problem is the lack of precise methods used in learning (Olpak, Baltaci, & Arican, 2018). According to Borovcnik (2017) often the teaching approach leads to the provision of formulas and regular application. This method makes some probability ideas unreachable by most students (Budgett & Pfannkuch, 2019). These difficulties occur as a result of the inability to acquire and apply mathematical skills and concepts, to reason, and to solve mathematical problems.

Based on the problems already mentioned above, it is important to study the mental structures that occur in the mind of students when studying concepts and solving certain problems. The process of knowledge formation occurs when a person develops a schema of reflective abstraction results that has been possessed in dealing with problems. The process of a person in constructing knowledge can be used as a basis for decision making when facing mathematical problems (Kurniati & As' ari, 2018). Reflective abstraction involves student activity directed at goals (physical and mental) and reflection (the ability to identify similarities in one's activities). Reflective abstraction results in the ability of students to anticipate the results of their activities.

2. METHOD

The current study used the exploratory research design, with the researcher as the key instrument. The data of this study was taken from fifth semester mathematics students. The study involved thirty participants, aged between 20-23 years. Based on the results of the written test, three participants were selected to be used as research samples using a purposive sampling technique (Sukestiyarno, 2020). The selection of the participants was based on certain criteria such as the one who can provide detailed information about mental mechanisms in solving continuous random variable problems; one who has scored good grades in the tests given; and the one who is able to provide complete information during interviews Furthermore, the participants who were used as research samples were named as the first participant, the second participant, and the third participant.

The researchers acted as key instruments of this research, besides using essay tests, and interview guidelines as additional instruments. These research instruments were in the form of test and interview questions to collect data on the thought process of solving problems. These instruments were validated by experts. Interviews with the three subject samples were conducted by researchers via Google Meet, separately for each subject sample and lasted about 45-60 minutes. The conversation was recorded to get a complete transcript of the interview. Interview questions were used to clarify the collected test answer sheets and strengthen the coding process (Creswell & Plano, 2018). Documentation of data was done in the form of written and video test results when doing test questions whose implementation was carried out through a zoom meeting. The data validity test in this study used triangulation of data collected through various sources, namely comparing the results of student data answers and in-depth interviews. Steps in data analysis included: 1) processing and preparing data for analysis. This step involved interviewing transcripts, scanning material, writing field data, and compiling lists into different types; 2) Reading the entire data. At this stage, the researcher wrote down a special note about the data obtained; 3) Data encoding. Coding was done by coding on answer sheets and interviews; 4) describing the data and present it in the narrative; 5) Data interpretation. Carried out by comparing the results of the study with information derived from literature or theory (Creswell, 2014).

3. RESULTS AND DISCUSSION

3.1. First Participant Analysis

The researcher analyzed the first participant's answer sheet with support from the interview results. Based on the results of the test answers, the first participant was able to give good answers related to the problems given. The following is an excerpt of an interview with the first participant.

Researcher (R) : "What did you think about when you first read the questions?"

First participant (FP) : "I read the problem first ma'am, then I saw that this problem is a continuous random variable."

Based on the interview results, when reading the questions, the first participant was able to identify the type of random variable given and tried to recall the definition of the density function. At this stage, the construction process carried out by the first participant was interiorization where he tried to understand the information and instructions involved, and read carefully the information about the questions and instructions. This suggests that first participant sought the best understanding before working on mathematical problems by reasoning and paying attention to important details related to the corresponding value to obtain a definition of the density function. Subsequently, he was involved in reading, understanding, and conveying it to mind, processing it into developing thoughts then pouring it into the doubles integral concepts and the probability density function. This activity reflects abstract thinking from an unstructured abstract to a developed abstract (Simon, 2020). After constructing internal processes in order to understand perceived phenomena, first participant coordinated the process by using the properties of the probability density function to find the value of the constant and used the calculation of the course integration. Previously possessed knowledge related to integrals needed to be used here. The first participant integrated it first against ‘y’ and then against ‘x’. Figure 1 presents the first participant 's answer.

If we know the combined density function

$$f(x,y) = \begin{cases} cx(1+3y^2); 0<x<2, 0<y<1 \\ 0 \quad ; \; x \, dan \, y \, others \end{cases}$$

based on the above function

a. value of c

$$c\int_0^2\int_0^1 (x+3xy^2)dy\,dx = 1$$

$$c\int_0^2 [xy+xy^2]_0^1\,dx = 1$$

$$c\int_0^2 (x+x)dx = 1$$

$$2c\int_0^2 \left[\frac{1}{2}x^2\right]_0^2 = 1$$

$$2c(2) = 1$$

$$4c = 1$$

$$c = \frac{1}{4}$$

Figure 1. First participant's answer in determining c (constant).

First participant further coordinated the value of c (constant) already obtained for the marginal density function of the continuous random variables X and Y to estimate whether the two random variables were stochastic-free as seen in Figure 2.

b. In your opinion, is the density function a statically free random variable? Explain!
Yes, because the product of the marginal product of X and Y is $f(x, y)$

$$f(x,y) = \frac{1}{4}x + \frac{3}{4}y^2x$$

$$f_1(x) = \frac{1}{2}x, f_2(y) = \frac{1}{2} + \frac{3}{2}y^2$$

$$f(x,y) = f_1(x). f_2(y)$$

$$= \frac{1}{2}x\left(\frac{1}{2} + \frac{3}{2}y^2\right)$$

$$= \frac{1}{4}x + \frac{3}{4}xy^2$$

∴ It is proven that the density function is a stochastic independent random variable

Figure 2. First participant's answer in determining stochastic-free random variables.

Although first participant did not describe how to get the marginal density function of X and Y on the answer sheet, which could be because the density function given was a stochastic-free random variable. To get a complete description related to the first participant's answer sheet in answering point b), he was called for an interview. Here is an analysis of the selective abstraction thought process of the first participant, which is evidence of solving the problem of distributing two continuous random variables shown in Figure 3 and Table 1.

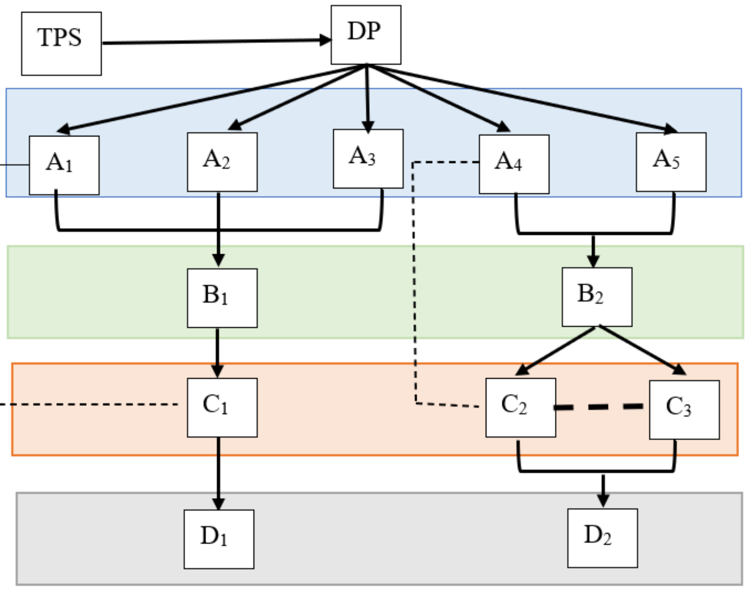


Figure 3. First participant's mental mechanism.

Figure 3 illustrates the first participant's thought process to get the desired solution, starting with actions by considering the concept to be used (A1), identifying the concept to be used (A2-A5); then the action was coordinated into a process (B1). At this stage the first participant thought of a way to get the value of the constant by coordinating actions A1, A2, A3 and the nature of the random variables were mutually free (B2) by coordinating actions A2 and A5. The object stage (C1, C2, and C3) occurred by encapsulating the values of constants and marginal functions from X and Y that had been processed previously. This became a reference to generalize (D1 and D2) in deciding on the right solution and scheme.

Table 1 Explanation of the flow of first participant's mental mechanism.

Code	Information	Code	Information
TPS	First Participant's Thinking process	C_3	Stochastic random variables known to set based on C2 and B2
DP	Distribution problem of continuous random variables	D_2	Generalizations based on C1, C2, C3, and D1
A_1	Considering the concept of double folding integrals	D_1	Generalizations based on C1, C2, and C3
A_2	Identification of continuous random variables	D_2	Generalizations based on C1, C2, C3, and D1
A_3	Identify the properties of the density function	□	Reflective abstraction indicator
A_4	Identification of marginal density functions	⟶	The displacement between reflective abstraction components (Directly)
A_5	Identify the definition of the stochastic-free random variables	------	The displacement between reflective abstraction components (Indirectly)
B_1	Coordination of A1, A2, and A3		Interiorization
B_2	Coordination of A2 and A5		Coordination
C_1	The setting gets a value of c (Constant) based on B1 and A1		Encapsulation
C_2	Setting get marginal density functions of X and Y based on B an A4		Generalization

Note: A= Action, B= Process, C=Object, D=Schema.

3.2. Second Participant Analysis

Second participant began to solve the problem by wisely analyzing the information in the problem and instructions, to see whether it identified the combined density function $f(x,y) = cx(1+3y^2); 0 < x < 2; 0 < y < 1$. It was clear that the second participant also sought to recognize the information and directions involved. The process of understanding information showed interiorization. Second participant underwent interiorization in solving the given problem. Furthermore, it is said that coordinates the process to get the value c (constant), using the density function properties that are $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$, as well as the integral concept of double folding to obtain the value of c. This statement shows that second participant through coordination between mathematical objects on the problem, namely the concept of integral and the concept of density function. In the results of his work, the second participant performed encapsulation by multiplying between $f_1(x)$ with $f_2(y)$ to obtain the initial hypothesis regarding two stochastic-free random variables. Eventually, the second participant performed a reversal to reverse its original structure as seen in Figure 4.

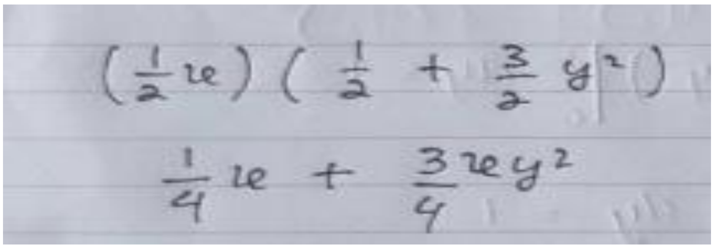


Figure 4. Second participant performs a reversal.

In this case the Second participant tries to double-check whether it is true that the initial conjecture about the arrangement to multiply between $f_1(x)$ with $f_2(y)$ to obtain the initial hypothesis of the two stochastic-free random variables obtained. Once convinced of the answer, second participant can find the relationship between the two marginal density functions and explain that they are two stochastic–free random variables, because $f(x, y) = f_1(x) . f_2(y)$, and concluded that jika $f(x,y) \neq f_1(x) . f_2(y)$ then the two random composers were said not to be stochastic-free or dependent. Figure 5 and Table 2 show the analysis of the thinking process of reflective abstraction second participant, in solving the problem of the distribution of two continuous random variables.

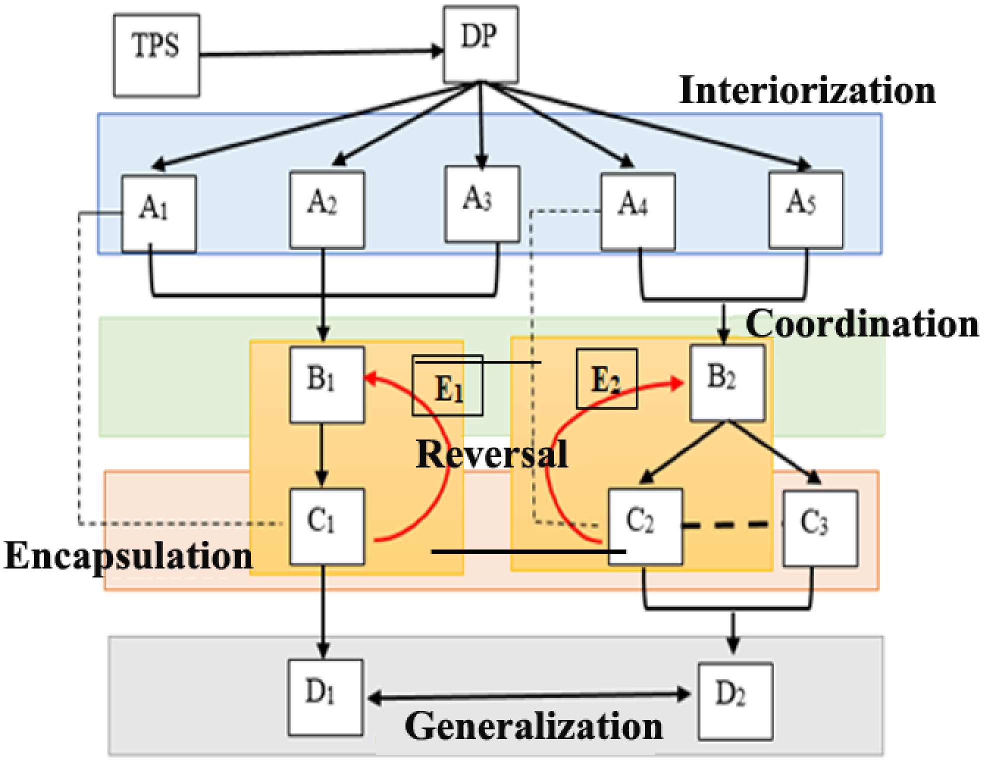


Figure 5. Second participant's mental mechanism.

Table 2 Explanation of the flow of second participant's mental mechanisms.

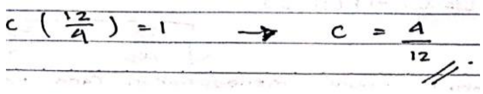
Code	Information	Code	Information
TPS	Thinking process second participant	D_2	Generalizations based on C1, C2, C3, and D1
DP	Distribution problem of continuous random variables	E_1	Reversal of C1 to B1
A_1	Considering the concept of double folding integrals	E_2	Reversal of C2 to B2
A_2	Identification of continuous random variables	□	Reflective abstraction indicator
A_3	Identify the properties of the density function	→	The displacement between reflective abstraction components (Directly)
A_4	Identification of marginal density functions	------	The displacement between reflective abstraction components (Indirectly)
A_5	Identify the definition of the stochastic-free random variables		Displacement for reversal
B_1	Coordination of A1, A2, and A3		Interiorization
B_2	Coordination of A2 and A5		Coordination
C_1	The setting gets a value of c (Constant) based on B1 and A1		Encapsulation
C_2	Settings get marginal density functions of X and Y based on B2 and A4		Generalization
C_3	Stochastic random variables know to set based on C2 and B2		Reversal
D_1	Generalizations based on C1, C2, and C3		

Note: A= Action, B= Process, C=Object, D=Schema.

Figure 5 illustrates the second participant's thought process to get the desired solution, starting with actions by considering the concept to be used (A1), identifying the concept to be used (A2-A5); then the action is coordinated into a process (B1), at this stage the second participant thinks of a way to get the value of the constant by coordinating actions A1, A2, A3 and the nature of the random variables are mutually independent (B2) by coordinating the actions A2 and A5. The object stage (C1, C2, and C3) occurs by encapsulating the values of constants and marginal functions from X and Y that have been processed previously. What distinguishes the second participant's way of thinking from the first participant is the reversal mental construction (E1 and E2). Based on this stage, it becomes a second reference for generalizing (D1 and D2) in deciding on the right solution and scheme.

3.3. Third Participant Analysis

Like first and second participant, the third participant also began trying to solve the problem by wisely reading the information given to the problem. Then third participant identified the properties of the density function used. In this case, third participant sought to understand the information and instructions in the question. The process of understanding such information manifests interiorization, which is evident in third participant's efforts to solve the given question. However, third participant was not precise in coordinating the process to get the value c (constant), using the density function properties of X that have been interpreted by third participant . Although in the end answer given was correct, but there was a visible discrepancy *Like value* $c = \frac{4}{12}$, then the next written $\frac{1}{4}$ as seen in Figure 6.



Yes, because the marginal multiplication of X and Y gives $f(x, y)$ is

$$F(x, y) = \frac{1}{4}x + \frac{3}{4}y^2x$$

$$f_1(x) = \frac{1}{2}x, f_2(y) = \frac{1}{2}x + \frac{3}{2}y^2$$

$$f(x, y) = f_1(x).f_2(y)$$
$$= \frac{1}{2}x.\left(\frac{1}{2} + \frac{3}{2}y^2\right)$$
$$= \frac{1}{4}x + \frac{3}{4}xy^2$$

it is proven that the density function above is a stochastic random variable

Figure 6 Third participant's answer mismatch

From Figure 6, and based on the results of the interview with the third participant, the process of solving the problem done by third participant tends to do the interiorization process. The statement is evident from the truth checking behind the information related to the question and coordination to make problem-solving conclusions mediated by all objects concerned. Interiorization is described as the stage in which students perform operations on lower-level mathematical objects. Here is an analysis of the third participant reflective abstraction thought process, in solving the problem of distribute two continuous random calamities shown in Figure 7 and Table 3.

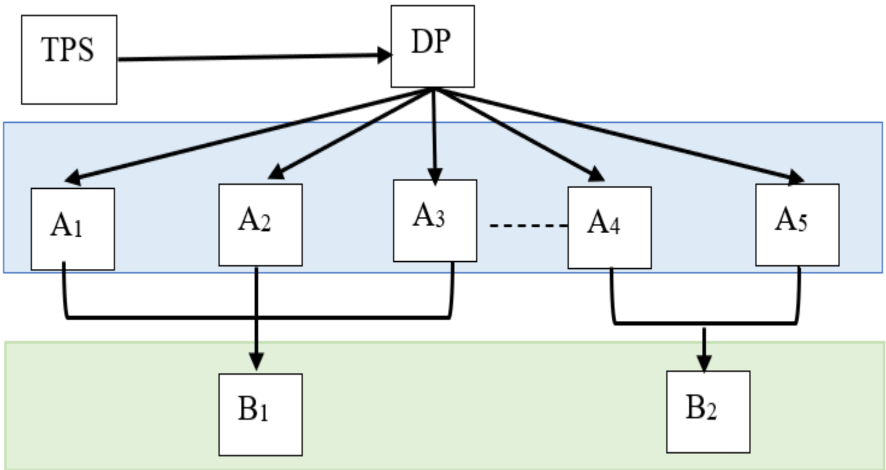


Figure 7. Third participant's mental mechanism.

Table 3 Explanation of the flow of third participant's mental mechanisms

Code	Information	Code	Information
TPS	Thinking process third participant	B1	Coordination of A1, A2, and A3
DP	Distribution problem of continuous random variables	B_2	Coordination of A2 and A5
A_1	Considering the concept of double folding integrals	(blue box)	Interiorization
A_2	Identification of continuous random variables	(green box)	Coordination
A_3	Identify the properties of the density function	□	Reflective abstraction indicator
A_4	Identification of marginal density functions	→	Identification of marginal density functions
A_5	Identify the definition of the stochastic-free random variables		

Note: A= Action, B= Process, C=Object, D=Schema.

Based on the test results and interviews, the first participant went through four components of reflective abstraction, namely interiorization – coordination – encapsulation – generalization; the second participant adopted five components of reflective abstraction thinking namely interiorization – coordination – reversal – encapsulation – generalization; and the third participant failed to solve the problem because the thought process in his mental mechanism had not yet manifested coordination, encapsulation and generalization. Coordination of actions turns into an activity (classification of actions) into one higher-level action. That is, as a result of his practice involved in activities, the student no longer needs to go through a sequence of actions; he could undo the outcome of that sequence. The first and second participants were both able to answer the given questions correctly even though the reflective abstraction components passed were different. The reversal component did not pass through by first participant and did not perform the algorithm process perfectly. Although first participant did not perform a reversal stage of its reflective abstraction thought process, it could draw correct and precise conclusions in explaining why a given density function is two stochastically free random variables. These findings are in line with previous research that stated that reversing concepts is often problematic for students. These findings support the opinion (Sopamena, 2018), that student way of thinking refers to the construction of reflective abstractions referred to as simple closed paths. The path that students walk is interiorization - coordination - encapsulation - generalization then coordination - encapsulation - generalization. Where in the results of this study, reversals were not carried out but students were able to solve the problem correctly.

In this study, the problem given to mathematics students was regarding continuous random variables. Based on the results of the interviews, mathematics students stated that solving the problem of continuous random variables was more difficult than discrete random variables, as experienced by the third participant. The difficulty was experienced by the third participant because integral concepts must be used to solve the problem of continuous random variables. Some student misconceptions make the topic of continuous random variables more difficult than discrete random variables (Kachapova, 2012). Based on the results and discussions, it was found that to improve student performance in mathematics learning, especially in courses such as probability theory, it can be started by paying attention to the process of reflective abstraction of prospective mathematics teachers. The construction of interiorization, coordination, encapsulation, generalization, and reversal should be examined in the process of developing a mathematical curriculum. In the process of learning mathematics, students' reflective abstractions can more easily appear and be known by using and expressing perceptions in learning (Wafiqoh, Kusumah, & Juandi, 2020). When facing the same problem, students will use different ways. The concepts that students already have are applied and any relationship between them is used in solving problems (García-Martínez and Parraguez (2017). Teachers need to design various activities so that students can accommodate the process of reflective abstraction in solving problems, such as giving feedback, implementing games, using computer programs, conducting class discussions, and so on (Simon, 2020). Feedback is also needed as a key role in promoting reflective abstraction (Mezhennaya & Pugachev, 2019). Other activities that can be designed to assist students in carrying out mental constructs that are considered lacking include the use of computers in learning, class discussions, and the provision of exercises (Borji & Martínez-Planell, 2019).

4. CONCLUSION

Reflective abstraction involves mathematics student activity directed at goals (physical and mental) and reflection (the ability to identify similarities in one's activities). The process of knowledge formation occurs when a person develops a schema of reflective abstraction results that he already has in the face of problems, so it is important to study the mental mechanisms that occur in the student's mind when learning certain concepts and solving problems. The mental mechanisms in question are interiorization, coordination, reversal, encapsulation, and generalization. This study describes the reflective abstraction process of three mathematics student participants in solving problems related to continuous random variables. The first participant went through four components of reflective abstraction: interiorization – coordination – encapsulation – generalization. The second participant adopted five components of reflective abstraction thinking, namely interiorization – coordination – reversal – encapsulation – generalization. The third participant failed to solve the problem because the thought process in his mental mechanism had not yet manifested coordination, encapsulation and generalization. Researchers found that the mental construction that the first participant had gone through could illustrate that the minimum mental construction that must be passed in solving the problem of continuous random variables is interiorization – coordination – encapsulation – generalization.

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