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Students' misconceptions and difficulties with learning limits and the continuity of functions at selected Rwandan secondary schools

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ABSTRACT

This study investigates students' difficulties and misconceptions with learning limits and explores the factors contributing to these misconceptions. The study used a mixed method approach, and the participants comprised 252 students and 21 teachers. Data collection methods included the Limits and Continuity of Functions Achievement Test, classroom observations, and focus group interviews. The study's findings revealed significant differences in the mean scores of the Limits and Continuity of Functions Achievement Test among different representations. It also identified that students' misconceptions stemmed from their limited exposure to active learning approaches and their tendency to rely on rote memorization. The study pointed out that teaching strategies, along with students' haste in providing responses, contributed to these misconceptions and learning difficulties. The fast-paced nature of teaching aimed at covering the syllabus requirements also played a role in creating knowledge gaps and reinforcing misconceptions. Through continuous professional development, this study recommends improving mathematics teachers' content, pedagogical knowledge of teaching and learning limits, and continuity of functions.

Contribution/Originality: This research elucidates students' challenges in understanding various representations of limits and continuity concepts, exploring associated misconceptions and causal factors, and proposing solutions for their mitigation. Its outcome will assist education stakeholders in strategizing to enhance the efficiency of teaching and learning calculus.

1. INTRODUCTION

The limits of functions concept dates back to Isaac Newton and Gottfried Leibniz in the 17th century when they developed the foundations of calculus. Newton and Leibniz used limits to analyze the behavior of functions at points of discontinuity and define the derivative of a function (Bos, 2000). In the 19th century, Augustin-Louis Cauchy extended the concept of limits, continuity, and derivatives of the functions of two or more variables (Sastry, 2006). In the 20th century, mathematicians made significant progress in the study of limits, particularly in relation to uniform convergence. They further refined the understanding of how functions approach their limits uniformly across their entire domain. Another important concept introduced was uniform continuity, which enabled the examination of how functions behave near points of discontinuity or singularities. These developments expanded the mathematical understanding of function behavior and paved the way for further advancements in analyses (Bressoud, Ghedamsi, Martinez-Luaces, & Törner, 2016; Debnath, 2004).

Limits play a significant role in teaching and learning calculus, such as solving integral calculus problems, determining function behavior and defining derivatives. While continuity assures the smoothness of functions, limits enable the assessment of values and determining convergence. Limits are used in the Fundamental Theorem of Calculus to compute the area under curves (Orton, 1983; Sealey, 2014). Different mathematicians have mentioned different approaches to teaching the concept of limits. Cauchy's approach focuses on the concept of a limit as the number that a function approaches as the independent variable approaches a given number. Heine's approach focuses on how the limit can be determined by examining the behavior of the function at a given number. Both approaches involve the same concept of a limit but emphasize different aspects. Cauchy's approach emphasizes the numerical value of the limit, whereas Heine's approach emphasizes the behavior of the function at the limit. Both approaches differ in their level of conceptual understanding but are closely related (Sinkiewicz, 2016).

Considering the role of limits in calculus, many students have an incomplete or inaccurate understanding of limits and the continuity of functions (Edwards, Dubinsky, & McDonald, 2005; Juter, 2006). For instance, if a graph has multiple discontinuities, it is difficult to determine the exact limit at each discontinuity (Nair, 2010). Students often confuse the limits of a function when using a graph or an image of the function. This confusion arises because the limits and image of point at a given function can often be seen on a graph (Alam, 2020; Cottrill et al., 1996). Researchers have distinguished between the concept image and the limit of a point at a given function. The limit can be determined by examining the behavior of the function at the point, whereas the concept image focus on the numerical value of the function at a given point.

1.1. Research Problem

There are several difficulties and misconceptions, including whether a function can reach its limit, the distinction between dynamic and static processes with limits, the erroneous assumption that the limit of a function is equal to the value of the function at a particular point, and the incorrect application of finding the limit when the limit of function tends toward infinity (Areaya & Sidelil, 2012; Bezuidenhout, 2001; Thabane, 1998; Thompson & Harel, 2021; Williams, 1991). The initial approach to teaching limits is the algebraic computation and properties rather than the concept of the limit itself. Several factors conflict with the formal definition at this stage. It is believed that students develop images of limits and infinity because of misconceptions about getting close, growing large, or continuing infinitely (White & Mitchelmore, 1996).

Understanding limits and continuity prepares students for more advanced study of differential and integral calculus, which are basic ideas in college- and university-level mathematics. According to Sofronas et al. (2015), many first-year university students cannot link concepts to skills, nor can they master calculus concepts or the fundamental skills of calculus. The abstract nature of calculus can result in a negative attitude toward mathematics and science at secondary and university levels. It can also lead to persistent poor performance in mathematics and a low rate of students enrolled in mathematics options (De Vera et al., 2022; Kunwar, 2021).

In 2014, the African Union initiated work to improve the quality of teaching and learning calculus across the continent (Bethell, 2016). They revised the curriculum, textbooks, exercises, and examinations to focus more on real-world applications, problem solving, and critical thinking. In 2016, they also developed and implemented a unified curriculum for teaching calculus across the continent, which emphasizes the use of technology to enhance learning. They have also reviewed and revised existing calculus programs across Africa to suit the needs of students (Ely, 2021). Additionally, before 2015, in the old Rwandan curriculum, which was a knowledge-based curriculum (KBC), different concepts of calculus, such as limits, continuity, asymptotes and derivatives, were taught in senior five (advanced level year 2). During the review of the KBC to a competence-based curriculum (CBC), the concepts were greatly emphasized and taught from senior four (advanced level year 1) to senior six (advanced level year 3) as the key concepts of calculus and mathematical analysis (REB, 2015).

1.2. Research Focus

This study focuses on the examination of the difficulties faced by senior four students in the limits and continuity of one real variable function in nine public schools in Burera and Gicumbi districts. Although students' difficulties with limits and continuity of functions have been widely researched, little research has been conducted on students' achievement across all the categories of the achievement test.

1.3. Research Questions

This study sought to answer the following questions:

1. Is there a difference in the Limits and Continuity of Functions Achievement Test scores by category?
2. What kinds of misconceptions do they present on the limits and continuity of functions?
3. What are the factors contributing to learners' misconceptions of limits and continuity of functions?

2. RESEARCH METHODOLOGY

2.1. General Background

This research employed the concurrent mixed methods approach, which entailed the collection and analysis of quantitative and qualitative data. The one group post-test only design was used for the quantitative component of the study. This means that a single group of student participants were only tested after treatment. The qualitative component of the study employed phenomenology, which entailed classroom observations and focus group interviews with students and mathematics teachers. The data from both was then integrated to arrive at an overall finding for a better understanding of the phenomenon being studied (Li, Sanders, & Frenkel, 2012).

2.2. Study Participants

The sample comprised 252 senior four students aged 16 and 17 with mathematics options, with 51.6% who were male and 48.4% who were female. Of the 21 mathematics teachers who were purposively sampled from nine public schools in the Gicumbi and Burera districts for this study, 16 were male and 5 were female, and 71.43% had at least a Bachelor of Science degree. These teachers had at least two years of teaching experience.

Table 1. Coverage of the limits and continuity of functions achievement test.

Category	Subtopic from which the test items were drawn	Maximum score per category
1	Finding limits using definitions	4
2	Concept of limit and continuity	7
3	Limit and continuity of a piecewise function	8
4	Limit and continuity for rational function (Removable discontinuity)	6
5	Sketching a graph and identifying the region where the function is continuous	7
6	Limits of point from graph and discuss its continuity	8
7	Real-world problems related to discontinuity	6
8	Algebraic method for finding limits of functions	4
	Total	50

2.3. Instrument and Procedures

Quantitative data was collected using the Limits and Continuity of Functions Achievement Test (LCFAT), while qualitative data was collected using the Reformed Teaching Observation Protocol (RTOP) and focus group interviews. The LCFAT had eight categories and was used to identify the areas of the LCFAT that students had difficulties with. The LCFAT was limited to the limits and continuity of polynomial, rational, irrational, and piecewise functions under the Rwandan mathematics secondary school syllabus for the senior four level. The question categories in the LCFAT are described in Table 1.

The RTOP and focus group interviews were used to identify the misconceptions and factors that influence students' understanding of limits and continuity of functions. The classroom setting, teachers' teaching methods, and students' behavior during lessons were observed to gain a better understanding of the limits and continuity of functions.

2.4. Validity and Reliability of Instruments

The instruments were validated for appropriateness of construct and content. Draft copies of the all the instruments were subjected to a critical review by four calculus university lecturers and three secondary school mathematics teachers and experts in the Testing and Evaluation Department of the University of Rwanda-College of Education. This was followed by a pilot test of the LCFAT with 40 senior four students, who were excluded from the study sample, to establish the reliability of the instrument. The reliability coefficient was 0.72, computed in SPSS, which shows a satisfactory internal consistency between the items and reliability (Li et al., 2012). The pretest of the LCFAT was immediately administered for the classroom observation of nine teachers. The interviews with 30 students and 21 teachers were conducted immediately after the post-test.

2.5. Data Analysis

Data analysis started by coding the marked scripts and questionnaires. Each script had a unique code assigned, which was determined by the position of their name in the official class roster. Afterwards, the quantitative data was analyzed using descriptive statistics in SPSS version 25.0. The qualitative data was analyzed using content analysis.

3. RESEARCH RESULTS AND DISCUSSION

This study sought to examine the differences in the Limits and Continuity of Functions Achievement Test scores by category and identify the misconceptions that students hold on limits and continuity of functions. The results and findings are presented in Tables 2, 3, 4 and 5.

Table 2. Descriptive statistics of the limits and continuity of functions achievement test scores by category.

Category	Mean	Std. deviation	Std. error
Category 1	0.206	0.5690	0.0358
Category 2	2.155	1.1202	0.0706
Category 3	2.341	1.4123	0.0890
Category 4	1.218	0.7909	0.0498
Category 5	2.381	2.2771	0.1434
Category 6	3.000	2.9444	0.1855
Category 7	0.649	0.8465	0.0534
Category 8	2.238	1.4445	0.0910

Note: Category 1 = Finding limits using definitions; Category 2 = Concepts of limits and continuity; Category 3 = Limits and continuity for a piecewise function; Category 4 = Limits and continuity for a rational function; Category 5 = Sketching a graph and deciding the region where the function is continuous; Category 6 = Limits of a point from a graph and discussing its continuity; Category 7 = Real-world problems related to discontinuity; Category 8 = Algebraic method for finding the limits of functions.

Table 2 shows a wide variance in the means and standard deviations of the LCFAT scores across the categories. Also, the minimum score for each category is the same, which is zero, unlike the maximum scores, which range from 2.0 to 12.0. Figure 1 displays a more visible variation of the mean scores for each variable.

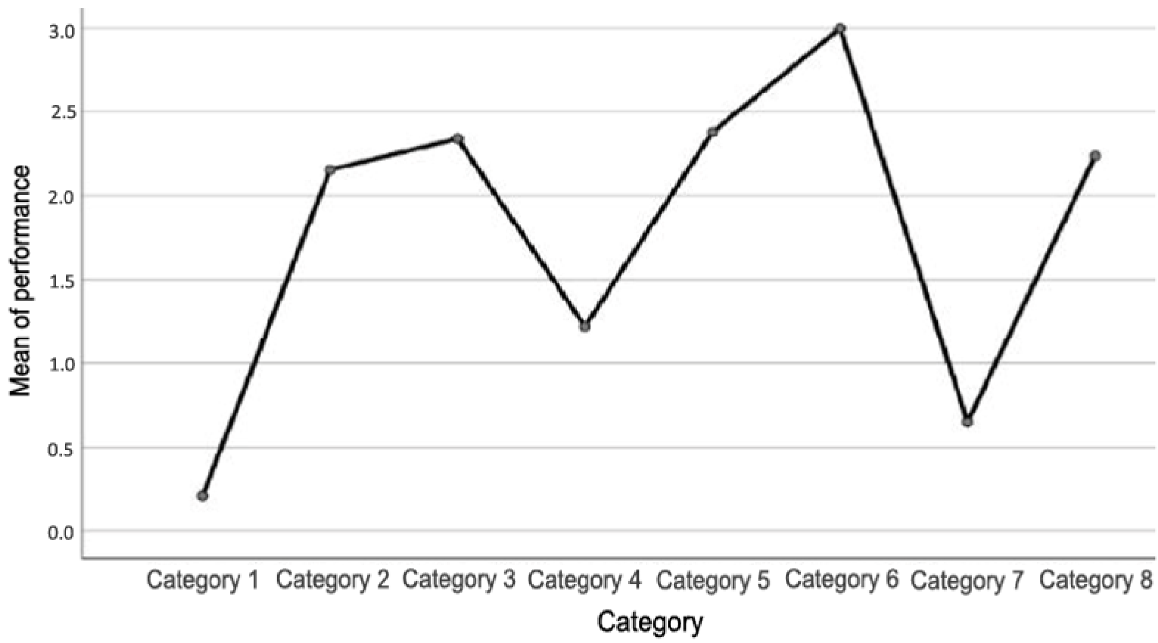


Figure 1. Mean plots of the LCFAT scores by category.

Figure 1 shows that category 6 has the highest mean score, followed by categories 5, 3, 8, 2, 4, 7 and 1. To test whether the variance in the scores is the same for each of the eight categories, we used Levene's test for homogeneity of variances, the results of which are shown in Table 3.

Table 3. Results of Levene's test for homogeneity of variances.

Performance		Levene statistic	df1	df2	Sig.
	Based on median	157.915	7	2007	0.000
	Based on median	108.504	7	2007	0.000
	Based on median and with adjusted df	108.504	7	866.441	0.000
	Based on trimmed mean	152.979	7	2007	0.000

Since Levene's test for homogeneity of variances shows a significant value less than .05, which implies a violation of the assumption of homogeneity of variances, reference was made to the Robust Test of Equality of Means instead (Pallant, 2005).

Table 4. Robust test of equality of means.

Performance	Statistic[a]	df1	df2	Sig.
Welch	204.942	7	845.963	0 .000
Brown–Forsythe	90.863	7	966.570	0.000

Note: a. Asymptotically F distributed.

However, a closer look at the Robust Test of Equality of Means in Table 4 shows that there are significant differences in the performance scores among the LCFAT categories. To ascertain the categories with significant differences, the post hoc test results were examined. Table 5 presents the analysis of the categories using the post hoc test to further examine the specific pairwise comparisons among the categories. It was used to determine which category's score differences are significant after considering the entire set of data, reducing the likelihood of chance findings.

Table 5. Post hoc test to identify statistically significant differences between categories.

(I) Category	(J) Category	Mean difference (I-J)	Std. error	Sig.	95% confidence interval (Lower bound)	95% confidence interval (Upper bound)
Category 1	Category 2	-1.9484*	0.1439	0.000	-2.385	-1.512
	Category 3	-2.1349*	0.1439	0.000	-2.571	-1.698
	Category 4	-1.0119*	0.1439	0.000	-1.448	-0.575
	Category 5	-2.1746*	0.1439	0.000	-2.611	-1.738
	Category 6	-2.7937*	0.1439	0.000	-3.230	-2.357
	Category 7	-0.4431*	0.1440	0.000	-0.880	-0.006
	Category 8	-2.0317*	0.1439	0.000	-2.468	-1.595
Category 2	Category 1	1.9484*	0.1439	0.000	1.512	2.385
	Category 3	-0.1865	0.1439	0.900	-0.623	0.250
	Category 4	0.9365*	0.1439	0.000	0.500	1.373
	Category 5	-0.2262	0.1439	0.767	-0.663	0.210
	Category 6	-0.8452*	0.1439	0.000	-1.282	-0.409
	Category 7	1.5054*	0.1440	0.000	1.068	1.942
	Category 8	-0.0833	0.1439	0.999	-0.520	0.353
Category 3	Category 1	2.1349 *	0.1439	0.000	1.698	2.571
	Category 2	0.1865	0.1439	0.900	-0.250	0.623
	Category 4	1.1230 *	0.1439	0.000	0.687	1.560
	Category 5	-0.0397	0.1439	1.000	-0.476	0.397
	Category 6	-0.6587 *	0.1439	0.000	-1.095	-0.222
	Category 7	1.6919 *	0.1440	0.000	1.255	2.129
	Category 8	0.1032	0.1439	0.997	-0.333	0.540
Category 4	Category 1	1.0119 *	0.1439	0.000	0.575	1.448
	Category 2	-0.9365 *	0.1439	0.000	-1.373	-0.500
	Category 3	-1.1230 *	0.1439	0.000	-1.560	-0.687
	Category 5	-1.1627 *	0.1439	0.000	-1.599	-0.726
	Category 6	-1.7817 *	0.1439	0.000	-2.218	-1.345
	Category 7	0.5689 *	0.1440	0.002	0.132	1.006
	Category 8	-1.0198 *	0.1439	0.000	-0.294	-0.583
Category 5	Category 1	2.1746 *	0.1439	0.000	1.738	2.611
	Category 2	0.2262	0.1439	0.767	-0.210	0.663
	Category 3	0.0397	0.1439	1.000	-0.397	0.476
	Category 4	1.1627 *	0.1439	0.000	0.726	1.599
	Category 6	-0.6190 *	0.1439	0.000	-1.056	-0.183
	Category 7	1.7315 *	0.1440	0.000	-1.056	2.168
	Category 8	0.1429	0.1439	0.976	-0.294	0.579
Category 6	Category 1	2.7937 *	0.1439	0.000	2.357	3.230
	Category 2	0.8452 *	0.1439	0.000	0.409	1.282
	Category 3	0.6587 *	0.1439	0.000	0.222	1.095
	Category 4	1.7817 *	0.1439	0.000	1.345	2.218
	Category 5	0.6190 *	0.1439	0.000	0.183	1.056
	Category 7	2.3506 *	0.1439	0.000	1.914	2.788
	Category 8	0.7619 *	0.1439	0.000	0.325	1.198
Category7	Category 1	0.4431 *	0.1440	0.000	0.006	0.880
	Category 2	-1.5054 *	0.1440	0.000	-1.942	-1.068
	Category 3	-1.6919 *	0.1440	0.000	-2.129	-1.255
	Category 4	-.5689 *	0.1440	0.002	-1.006	-0.132
	Category 5	-1.7315 *	0.1440	0.000	-2.168	-1.295
	Category 6	-2.3506 *	0.1440	0.000	-2.788	-1.914
	Category 8	-1.5887 *	0.1440	0.000	-2.026	-1.152
Category 8	Category 1	2.0317 *	0.1439	0.000	1.595	2.468
	Category 2	.0833	0.1439	0.999	-0.353	0.520
	Category 3	-.1032	0.1439	0.997	-0.540	0.333
	Category 4	1.0198 *	0.1439	0.000	0.583	1.456
	Category 5	-.1429	0.1439	0.996	-0.579	0.294
	Category 6	-.7619 *	0.1439	0.000	-1.198	-0.325
	Category 7	1.5887 *	0.1440	0.000	1.152	2.026

Note: * indicates the significance level with a p-value less than 0.05.

Table 5 highlights the statistically significantly differences between the mean scores for four categories—finding limits using definitions (Category 1), limits and continuity for rational function (Category 4), the limits of a point from a graph and discussing its continuity (Category 6), and the real-world problems related to discontinuity (Category 7)—and the rest of the categories. The mean scores for the concept of limits and continuity (Category 2) and limits and continuity for a piecewise function (Category 3) are statistically different from four other categories, namely finding limits using definitions (Category 1), limits and continuity for rational function (Category 4), limits of a point from a graph and discussing its continuity (Category 6), and real-world problems related to discontinuity (Category 7). Category 4, like category 1, differs significantly from all the other categories. The mean score for sketching a graph and deciding the region where the function is continuous (Category 5) differs significantly from the scores for the finding limits using definitions (Category 1), limits and continuity for rational function (Category 4), sketching a graph and deciding the region where the function is continuous (Category 5), limits of a point from a graph and discussing its continuity (Category 6), and real-world problems related to discontinuity (Category 7). The algebraic method for finding limits of functions differs significantly from finding limits using definitions (Category 1), limit and continuity for rational function (Category 4), limits of a point from a graph and discussing its continuity (Category 6), real-world problems related to discontinuity (Category 7), and the algebraic method for finding the limits of functions (Category 8).

Table 6. Students' misconceptions and difficulties related to limits and continuity of functions.

Category	Example	Common difficulties	Type of difficulties
Finding limits using definitions	Show that $\lim_{x\to 5}\frac{x^3-5x^2+x-5}{x-5}=26$	The majority of the students use the algebraic method instead of the definition of limits.	Conceptual understanding that there is a deficiency in the understanding of the definition of limits.
Limit and continuity of a piecewise function	Discuss the continuity at x = 5 $g(x)=\begin{cases}2x+1, x\geq 5\\ 3x-4, x<5\end{cases}$ f(x)= $\begin{cases}4x-3, & x\neq 5\\ 15, & x=5\end{cases}$	Certain students struggled to discern the left and right limits at x = 5. While some could identify the limit, they faced challenges linking it to the function's domain.	Procedural understanding where there is insufficient knowledge on conducting the steps correctly. Systematic understanding where the majority of the students do not connect procedural with conceptual knowledge to reach a conclusion about the continuity at the point.
Limit and continuity for rational function (Removable discontinuity)	Discuss the continuity of $h(x)=\frac{x^3-5x^2+x-5}{x-5}$	While removing the indeterminate form $\frac{0}{0}$, 147 students applied L'Hopital's rule, which requires skills on derivatives. Factorization skills are missing.	Conceptual understanding where knowledge related to factorization is insufficient.
Sketching a graph and deciding the region or point where	Sketch the following function and decide which region it is continuous f(x) = $\sqrt{2x-6}$	Most students sketched $\sqrt{x}$ and x = 3 instead of $\sqrt{2x-6}$ due to a	Procedural understanding where students could not do all steps properly. Also, there were conceptual

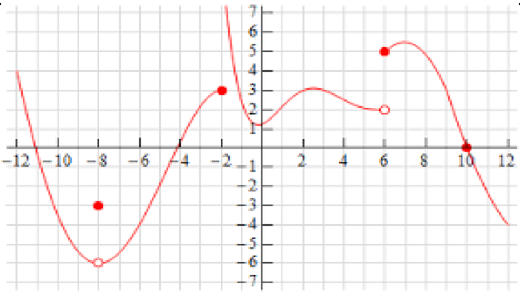
Category	Example	Common difficulties	Type of difficulties
the function is discontinuous		rough method of sketching functions. Also, many students solved the function instead of sketching.	errors in differentiating functions and equations.
Limits of points from a graph and discussing its continuity	 Discuss the continuity at x = -8, x = 10 and x = 6	The majority of the students confused the image of the point and its limit from the graph. They expressed that $\lim_{x \to a} f(x) = f(a)$	There is systematic misunderstanding, where students do not connect procedural knowledge to conceptual knowledge.
Real-world problems related to discontinuity	F(t) is the function that represents the total mass of the universe over time, measured in years since the big bang, t goes from -1000,000 to 1000,000. Assuming that before the big bang no matter was present, the universe was suddenly created. Decide and explain if it is: a. Removable discontinuity b. Jumping discontinuity c. Infinite discontinuity d. Continuous	Most of the students cannot decide whether it is continuous or discontinuous.	Conceptual understanding where there is inadequate understanding of the concept of continuity.

Table 6 presents the common difficulties for each category and the type of difficulties that the students presented for each category. The common difficulties and misconceptions were traced to the level of conceptual and procedural understanding. Students used the algebraic method to find the limits using definition instead of the definition of limits. Students had difficulties following the correct steps in identifying limits. In addition, some students simply had an insufficient understanding of factorization. Students misconceived functions for equations. On a sketched graph, students could not distinguish between an image of a point and its limit. Students could not figure out whether a real-world problem represented a continuous or discontinuous function.

3.1. Observation Data

Table 7 summarizes the observation checklist used during the classroom observations, and Table 8 presents the factors that influence misconceptions.

Table 7. Observation themes and subthemes using RTOP.

Main themes	Subthemes
Lack of multiple representation	• Presentation lacks clarity and concreteness • Focus on one presentation (Algebraic approach)
Lack of cooperative learning	• A variety of means and media were used less often by students to communicate their ideas
Lack of creative and critical thinking	• Conceptual understanding is less promoted in the lesson • The use abstraction (e.g., symbolic representation, theory building) was encouraged when appropriate
Lack of active learning	• Students were less reflective about their learning • It was highly valued to have intellectual rigor and constructive criticism and to challenge ideas

Table 8. Factors influencing students' misconceptions, difficulties and errors in learning limits and continuity of functions.

Source of difficulties and misconceptions	Explanations
Superficial understanding	• Students focus on the memorization of the rules without understanding them
Instructional method	• Lesson design may create misconceptions if the teacher cannot match pedagogical knowledge and content knowledge • Sufficient examples are not given • Lack of different methods of presentation of the concepts
Insufficient knowledge of the basic concept	• Basic knowledge of factorization • Basic knowledge on domain of function
Carelessness	• Students do not reflect on their answers, and they do not verify or check if their answers make sense
External limitations	• Too much syllabus to complete • Different students' abilities • Large number of students • School activities

The factors responsible for the difficulties and misconceptions are traced back to students' inadequate exposure to a variety of active learning activities and the lack of a variety of learning approaches that could meet students' learning needs and preferred learning styles. Students did not exercise sufficient critical thinking abilities, and this was due to the absence of creative activities in the institutional design. Table 8 shows that students are accustomed to memorization, which is not the same as deep learning. Also, the instructional strategies did not address students' learning needs. Students displayed an insufficient knowledge of factorization and domain of functions. In addition, students were in haste to submit their answers and responses to the questions. Therefore, they did not sufficiently verify their responses. Teachers are nervous about completing the required lesson content as prescribed by the syllabus due to the high number of topics compared to the contact time, challenges resulting from large classes, and the widely varying levels of students' skills and previous knowledge.

4. DISCUSSION OF THE RESULTS

Based on the lowest mean scores, the students' main difficulties are in the finding limits using definitions (Category 1) and the solution of real-world problems related to discontinuity (Category 7), while determining the limits of a point from a graph and discussing its continuity (Category 6) was the easiest category. Expectantly, this result portrays a generally low mean score in areas of mathematics that reflects relatively lower order thinking and reasoning skills. The minimum score of zero for each category indicates an insufficient level of previous knowledge of the topics, which could contribute to the unsatisfactory scores generally. Also, the wide range in standard deviations (0.5690–2.9444) shows that the students were significantly different in their skill and achievement levels. This observation is confirmed by the significant differences between and among categories, and generally significant differences in the scores between categories. The low and same minimum scores for each category could be due to insufficient preparation from previous grade levels. This means that teachers should include purposeful activities in their daily lessons to raise students' achievements through higher order questions and engagements.

In addition to the sources of misconception and difficulties represented in Table 8, there are intrinsic sources of misconception and difficulties in mathematics that cannot be avoided, such as misunderstanding the language of mathematics; mathematics uses its own language, and students can easily get confused if they are not familiar with the specific terms associated with the subject. During learning, the terms 'tends to' or 'approach' influence students' understanding of the limits of functions (Denbel, 2014). Poor problem-solving skills means that students may have difficulty breaking down problems into smaller steps and understanding the relationships between them. Poor memorization skills will lead to students forgetting rules and principles associated with mathematics, resulting in errors. Difficulty transferring skills and knowledge across different contexts means that students may understand a concept in one context but not in another (Sotos, Vanhoof, Van den Noortgate, & Onghena, 2007).

4.1. Discussion on Using a Formal Definition of Limits (Category 1)

In Table 2, the mean performance for category 1 is 0.206, which is extremely low. This shows that there is a huge challenge in finding limits by using definitions. According to many teachers, students have difficulty understanding the definition. Most students used an algebraic approach instead of a definition of limits or epsilon delta.

During the classroom observations, students asked several questions regarding notations and connections between ε and δ. One student asked, "Where do ε and δ come from?" When reading the definition, other students inquired if there was a relationship between ε and δ, L and a, and y and x. Additionally, they believed that ε, L, and y were related, but did not know how ε, δ, a, and x were related. Some noted that students did not understand how to interpret the absolute value and inequalities of the Cauchy approach to the definition of limits algebraically and geometrically. The majority of studies claim that students have difficulty understanding the Cauchy definition of limits when learning calculus, as evident in previous research (Mahmudov & Mahmudov, 2013). For students, Cauchy approaches are rare because they are used only for problems that can be solved more intuitively (Fernández, 2004; Seager, 2020).

It's not only the students who face difficulties handling Cauchy's definition of limits; the teachers faced the problem of handling inequalities and the relation of ε to δ (Cornu, 1991).

4.2. Discussion on the Concepts of Limits and Continuity (Category 2)

Some observed that on the concepts of limits and continuity, the performance of the respondents is improved compared to category 1. The mean performance for this category is 2.155, as the majority of the respondents answered the true or false answers correctly but stated that it was difficult. During the explanation of their answers, some students presented cognitive conflicts related to false arguments on the concepts of limits and continuity.

- The existence of a limit at a point implies continuity at that point.
- The function must be defined at that point for the existence of the limit of that function at that point.

Some students agreed that the function must be defined at that point to be continuous, which is a misconception. Continuous function at point implies that the function is defined at that point, but the function defined at a point does not imply a continuous function. For example, the piecewise function may be a counter example of what they reported.

$$\begin{cases} 4x - 3, & x \neq 5 \\ 15, & x = 5 \end{cases}$$

Is the function f above continuous at x = 5?

The function defined at $x = 5, f(5) = 15$, i.e., the function above is defined at $x = 5$

$$\lim_{x \to 5} 4x - 3 = 4(5) - 3 = 17$$

Since the limit of f(x) as x approaches 5 is 17, and the value of f(x) at x=5 is 15, these two values are not equal. Therefore, the function f(x) is not continuous at x = 5.

Cognitive conflict may arise in mathematical situations when conflicting conceptions are simultaneously evoked. In interviewing the students, some questions were aimed at causing cognitive conflict to determine if they were sure of their responses. Only 30 students out of 252 who were interviewed showed confusion regarding the existence of limits alongside the existence of continuity. Difficulties related to the relationship between a limit and continuity reflect improper mental representations. To meaningfully relate the concepts mentioned above, students should have a mature understanding of these concepts. In addition, some teachers mentioned that it is sufficient for a function to be defined at a point to be continuous. This shows a content knowledge gap in their understanding of the continuity function.

During the observation, many activities provided by the teachers did not focus on all the conditions of the continuity of function. Most questions were based only on the existing limits, and students lost concentration on the image of the point of a function. This result is supported by previous findings of researchers, who agree that there is confusion regarding the formation of a definition of continuity (Areaya & Sidelil, 2012; Ashlock, 2019; Bezuidenhout, 2001; Budak & Ozkan, 2022; Maharajh, Brijlall, & Govender, 2008).

4.3. Discussion on Limits and Continuity of Piecewise Functions (Category 3)

The mean performance on the limit and continuity of a piecewise function was 2.341, which is good compared with the other performance of other categories. In addition to this performance, many students showed a gap in connecting theoretical to dynamic concepts on piecewise functions. According to some students, continuous functions can be defined by using only one equation. The polynomial function is as follows:

$$f(x) = x^2 - 3x + 3$$

It was easier for many students to say that the function is continuous because every x has the value y, but given a piecewise function such as $f(x) = \begin{cases} 2x+1, & x \geq 5 \\ 3x-4, & x < 5 \end{cases}$, many students mentioned the left and right limits but did not indicate whether the x value was approaching the domain, which reflects the decision of continuity or discontinuity, but some were challenged with finding the limit from the left or right of a point. Instead of determining the limit, they solved each subdomain.

During an interview, students mentioned that it was not a single function, and their responses showed that there is a gap in the knowledge of sketching piecewise functions and finding the domain. According to different studies, students who do not understand the concept of domain functions may have difficulties understanding piecewise functions (Kratsios & Zamanlooy, 2022; Triutami, Hanifah, Novitasari, Apsari, & Wulandari, 2021).

Piecewise functions are continuous on an interval in their domain if their subfunctions are continuous along their respective intervals (subdomains) and the boundaries between the subdomains are continuous.

4.4. Discussion on Limits and Continuity of Rational Functions (Category 4)

For the category of limits and continuity for rational functions (removable discontinuity), the mean is 1.218 and the standard deviation is 0.7909. These unsatisfactory results are influenced by student's lack of connecting appropriately to the continuity of rational functions, limits, domain, asymptotes, and the image of the rational function.

$$f(x) = \frac{x}{x-1}$$

It was easier for students to determine that this function is not continuous at $x = 1$ because it is outside the domain, and as:

$x \to 1^+, f(x) \to +\infty$, but given $f(x) = \frac{x^3 - 5x^2 + x - 5}{x-5}$, it was difficult for students because they simplified the function as

$f(x) = x^2 + 1$ and wrote $f(5) = 26$, which is incorrect. This shows the knowledge gap on the role of the domain and vertical asymptote for rational functions. This result is supported by the work of other researchers who studied rational functions (Nair, 2010).

4.5. Discussion on Sketching Graphs and Determining the Region or Point of Discontinuity of the Function (Category 5)

There is good performance in this category, which concerns sketching a graph and determining the region or point of discontinuity of the function, with a mean of 2.381 and a standard deviation of 2.2771. In addition, students who perform well have difficulties sketching functions, such as piecewise, irrational, rational and absolute values, and showing the region of continuity of the functions was challenging for many students. There were students who demonstrated a mismatching of the domain of the definition and the continuity of a function and confusion in sketching piecewise and irrational functions.

During the interview, students explained how to draw parabolic functions, such as $f(x) = x^2 - 3x + 3$, but the irrational function was difficult for them. Many students simplified the irrational functions instead of sketching, which shows the knowledge gap regarding the role of the domain on sketching functions, and there is also a gap in differentiating functions and equations. Given $f(x) = \frac{x^2 - 10x + 25}{x - 5}$, the majority of the students simplified it before sketching, and sketched $f(x) = x - 5$, which is wrong because this function has a vertical asymptote at x = 5.

During the classroom observations, many teachers focused on the rough sketching method, which is memory-based graphics sketching rather than a pointwise approach to drawing a graph. Many students claimed that the pointwise method requires a long step to the sketch function, which bored them. It was observed that students who could not determine the domain of the function had incorrect graphs.

4.6. Discussion on Finding Limits of Points from a Graph and Discussing Their Continuity (Category 6)

Based on the results in Table 2, there was good performance in this category compared to the other items, with a mean of 3.000 and a standard deviation of 2.9444. Despite this good performance compared to the other categories, there was a misconception of differentiating the image and the limit of functions at a point. Some students mentioned that $\lim_{x \to a} f(x) = f(a)$, but this is not always true. It will be true when the function is continuous at $x = a$. Looking at the answer sheets, many students gave the correct responses on the point of discontinuity from the graph, even if they missed evaluating the limit from a graph, which shows that analytical approaches to semiotics are lacking when compared to graphical representations. Through visualization, students decided the continuity or discontinuity of a point from the graph but had difficulties finding the limit or an image of that point. It was observed that there was a gap in algebraic presentation from graphical interpretations.

4.7. Discussion on Real-World Problems Related to Discontinuity (Category 7)

Based on the performance mean, it was observed that this category was among those categories with poor performance, with a mean of 0.649 and a standard deviation of 0.8465. This shows that most students did not understand the concepts of continuity or discontinuity in real-world problems. Not only the concept of limit and continuity, but many studies also showed that students had a strong tendency to exclude real-world questions in mathematics (De Corte, Verschaffel, & Greer, 2000).

The community of mathematics educators places strong emphasis on teaching mathematics through linkages to the real world. According to researchers, students who participate in integrated learning situations also must apply what they have learned in a variety of circumstances. In addition, research has demonstrated that when students encounter challenging objects, they disregard realistic considerations and real-world information (Gainsburg, 2008; Stacey, 2015). An experimental classroom setting that excludes real-world knowledge can create the negative belief among students that school arithmetic real-world problems are tricky.

4.8. Discussion on Finding Limits using the Algebraic Method (Category 8)

The findings in Table 2 show that there was good performance in this category, with a mean performance of 2.238 and a standard deviation of 1.4445. This good performance is supported by many algebraic exercises in calculus textbooks and teaching methods, which strongly promote an algebraic approach rather than geometric and other mathematical representations. The majority of students were motivated when solving limits and continuity of algebraic expressions. In certain situations, students may disregard the benefits of other approaches if they are taught to use only algebraic principles. This result is supported by different studies which show that many teachers focus on the algebraic approach, which leads students to ignore other methods of understanding the different concepts in calculus (Donmez & Basturk, 2010; Mastorides & Zachariades, 2004).

5. CONCLUSION

This study aimed to investigate variations in test scores for the Limits and Continuity of Functions Achievement Test among different categories. It also aimed to delve into the challenges and misunderstandings encountered by students when learning about limits and continuity, as well as identify the factors contributing to these misconceptions. The results show significant disparities in the scores achieved in the Limits and Continuity of Functions Achievement Test across various categories. Most students encountered difficulties in tasks such as determining limits using the formal definition, graphing functions accurately, identifying continuous regions or points, calculating limits from a graph, and assessing whether a function exhibits continuity at a specific point. Additionally, students faced challenges when applying the concepts of limits and continuity to real-world scenarios, often struggling to discern whether functions are continuous or discontinuous in these contexts. It's worth noting that the difficulties that students face are not uniform; the most pronounced issues were observed in tasks related to determining limits using the formal definitions and applying the concept of discontinuity to real-world situations.

The research revealed that students faced challenges and misunderstandings in grasping the concepts of limits and continuity in functions. These difficulties stem from their limited understanding of how limits, domains, asymptotes, and the continuity of functions at specific points are interconnected. Their misconceptions are rooted in a somewhat simplistic view of mathematics. The students' misconceptions primarily arose because they had not been exposed to active learning methods, which would have provided them with opportunities for engaging and hands-on learning experiences. Another contributing factor to their struggles was the tendency to rely on rote memorization. While teaching strategies played a role in these misconceptions and difficulties, the students' haste in providing responses to the questions also contributed to errors. Additionally, the rapid pace of teaching aimed at covering the entire syllabus added to gaps in their understanding and fueled misconceptions.

6. IMPLICATIONS AND RECOMMENDATIONS

- Curriculum enhancement: The results of the study point to the necessity to review the mathematics curriculum and provide teachers adequate time to explain the connections between ideas such as limits, domains, asymptotes, and continuity of functions. It might be advantageous to teach these subjects in a more integrated way.
- Pedagogical changes: Teachers should consider active learning techniques when developing their lesson plans. With the help of these strategies, students can participate in worthwhile activities that encourage a deeper understanding of mathematical ideas. Teachers should be encouraged to use active learning strategies to make math more interesting and meaningful, such as group discussions, problem-solving activities, and real-world applications.
- Critical thinking: Teachers should promote critical thinking and problem-solving abilities rather than relying on rote memory. Students' conceptual knowledge of mathematics should improve with this change.

- Balancing pacing: Teachers should be encouraged to take a balanced approach to their lessons so students can explore mathematical ideas in more depth, decreasing the need to quickly cover the curriculum. Investment should be made in programs that provide educators with effective teaching techniques, particularly those that aim to overcome misconceptions and promote a deeper understanding of mathematics.
- Student assessment and feedback: Assessments should place more emphasis on conceptual understanding rather than merely fact recall. Students may be encouraged to pay closer attention to the material due to this modification. Quick motivational feedback a can be reinforced by educators.
- Supporting struggling students in mathematics: Support networks should be established for learners who have trouble understanding mathematical ideas by offering extra materials, tutoring, or specialized training as needed.
- Parents and community involvement: Encourage a good attitude toward math by involving parents and the community in spreading the word about the value of mathematics education.

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