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INFLUENCES OF NON-LOCALITY ON THE ELASTIC WAVE SURFACES IN ELASTIC MEDIA

K. L. Verma

Department of Mathematics, Career Point University Hamirpur (HP) India.
Email: klverma@netscape.net Tel: +919418090952

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ABSTRACT

Classical continuum theories restrict the response of the continuum stringently to local actions, thus these theories are not capable to explain some phenomena precisely where the length scales are often sufficiently short as in nanostructures where it is required to consider the small length scales. This paper within the framework of nonlocal elasticity is concerned with the study of wave-surface features in nonlocal elasticity for cubic crystals. The nonlocal Christoffel equation of wave motion is derived and dispersion relations are obtained. The present model predicts some notable features of the dispersion relations in cubic crystals in comparison with classical local model. By considering the wave and slowness surfaces in [100], [110], and [111] planes of cubic crystals a perceptible change is observed with nonlocality parameter. In nonlocal theory longitudinal and transverse waves, become dispersive and influenced by non-locality parameter, whereas theses waves are non-dispersive in its counterpart classical theory (local theory). It is found that phase and group wave velocities for longitudinal and transverse modes are influenced by the nonlocality parameter only when its value is greater than 0.001. Numerical calculation for crystals Silicon (Si), Aluminum (Al), Copper (Cu), Nickel (Ni), Gold (Au) are carried and found that velocities of longitudinal and transverse waves continuously decreases with increases of non-locality parameter. Polar diagram and wave's surfaces for phase and group velocities (m/s) of longitudinal and transverse and slowness surfaces are represented graphically in nonlocal elasticity.

Contribution/Originality: This study originates a generalized *nonlocal Christoffel equation* in the nonlocal theory of elasticity. Influences of nonlocal parameter (ε) on the wave's spectrum, anisotropy factor, slowness surfaces in various directions are investigated. The results obtained are exhibited in the tabular forms and represented graphically considering cubic crystals.

1. INTRODUCTION

In the conventional continuum mechanics, linear theory of elasticity is inherently size independent and predicts no dispersion and is valid only for small wave numbers. Elastic strain, the stress and the elastic strain energy of defects are singular at the imperfection line. Undoubtedly, if one makes use of classical elasticity within the imperfect region, then such unphysical singularities then the penalty has to be compensated. Because of such limitations, we need to consider the small length scales such as lattice spacing between individual atoms, grain size, the nonlocal elasticity theory pioneered by Edelen and Laws [1]; Edelen, et al. [2]; Eringen and Edelen [3]; Eringen [4] and Eringen [5]; Eringen [6] which state that the local position at a point is influenced by the action of all particles of the body. Edelen [7] published a treatise in which he gave a rigorous comprehensive analysis of the foundations of nonlocal theories.

In classical (local) elasticity, several researchers Miller and Musgrave [8]; Musgrave [9]; Farnell [10]; Brugger [11]; Musgrave [12]; Buchwald and Davis [13] and Mielnicki [14] in the past splendid introduction of the fundamental concepts is explained and studied the wave surfaces. Philip and Viswanathan [15] studied the behavior of the sections of the inverse velocity surfaces and found that a large number of cubic crystals exhibit cuspidal edges for the sections of energy surfaces along the (100), (110) and (111) directions. Narasimha and Viswanathan [16] studied elastic wave surfaces for the (111) plane of cubic crystals. Since all these studies are in accomplished in traditional classical continuum mechanics models, which are scale free or size-independent, and its application to extended wave limit according to the atomic theory is not capable to explicate the small nanoscale size effect Gurtin and Murdoch [17]; Gleiter [18]; Lim, et al. [19]. Consequently, properties which are associated with nanostructures like lattice spacing between atoms, grain size, surface stress, etc., must be taken into consideration in any of the classical continuum models to study the requirement of size-dependence and which is applicable to micro and nano structures. Recently, authors Khurana and Tomar [20]; Dilbag, et al. [21]; Kaur, et al. [22] studied waves problems microstretch solid, micropolar elastic solid half-space, and with voids in the context of nonlocal theory. Slowness is defined as the inverse of velocity and slowness surfaces, by means of Christoffel equation for the wave propagation in elastic media, displays many interesting features as in Buchwald and Davis [13]; Lin, et al. [23]; Fein and Smith [24]. Slowness surface has an vital physical significance as a succinct graphical representation of the variation of all types of velocity with respect to direction of the slowness vector and is used as a pictographic to explain. Slowness surface are two-dimensional entities in three-dimensional space. Studies of elastic waves in such simple and mostly isotropic systems are widely available in the books [25-30]. Verma [31] studied the thermoelastic slowness surfaces in anisotropic media with thermal relaxation in the local generalied thermoelasticity.

In the present work, due to the establishment of the nonlocal theory, the aspects of wave quantities required in constructing wave fields propagating elastic media are calculated as a function of the slowness vector or of its direction called the wave normal. Based on the nonlocal theory of elasticity by Eringen, analysis of some interesting wave-surface features are studied for an elastic materials of cubic symmetry. Longitudinal and transverse waves, become dispersive but non-attenuating and influenced by non-locality parameter in this nonlocal theory, whereas theses waves are non-dispersive in its counterpart classical continuum mechanics theory (local theory). Wave and slowness surfaces are studied in [100], [110], and [111] planes of cubic crystals. It is found that phase and group wave velocities for longitudinal and transverse waves are affected only when the magnitude non-locality parameter is greater than or equal to 0.001and decreases with increases of non-locality parameter. Phase and Group velocities (m/s) of longitudinal and transverse wave's surfaces polar diagram of phase velocity (m/s) and slowness surfaces are also represented graphically in nonlocal elasticity materials for Silicon(Si), Aluminum (Al), Copper (Cu), Nickel (Ni), Gold (Au).

2. NONLOCAL ELASTICITY THEORY

Recognizing an Eringen-type nonlocal differential model [5] the stress may be associated with the displacement in the analogous case of nonlocal elasticity. The integral constitutive relations can be represented in a linear differential form as an Eringen type differential model for the nonlocal elastic media as:

$$\left(1-\varepsilon^2\nabla^2\right)\sigma_{ij}=\sigma_{ij}^{\ell} \tag{1}$$

Here recognizing an Eringen-type nonlocal differential model [5] the stress may be associated with the displacement in the analogous case of nonlocal elasticity. In the Equations 2 σ_{ij} and σ_{ij}^{ℓ} are nonlocal and local stresses, respectively; $\varepsilon\left(=e_0a\right)$ is the nonlocal parameter wherein a is an internal characteristic length (lattice parameter, granular size or molecular diameters) and e_0 is a material constant evaluated by the experiment; ∇^2 is Laplacian operator.

The basic constitutive equations of linear, homogeneous of nonlocal elastic solid are given as:

Equations of motion

$$\sigma_{ij,j}+\rho f_i=\rho\ddot{u}_i \tag{2}$$

$$\sigma_{ij}\left(x\right)=\int_V \alpha\left(\left|x-x'\right|\right)\sigma_{ij}^{\ell}\left(x'\right)dV\left(x'\right) \tag{3}$$

Stress-strain relations

$$\sigma_{ij}=c_{ijkl}e_{kl}\,,\quad i,j,k,l=1,2,3, \tag{4}$$

Strain-displacement relations

$$e_{kl}=(u_{k,l}+u_{l,k})/2 \tag{5}$$

Where V is the region occupied by the body, $\alpha\left(\left|x-x'\right|\right)$ is nonlocal kernel function and $\left|x-x'\right|$ denotes the distance between the reference point x and any neighbor point x' in the continuum body; the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$c_{ijkl}=c_{klij}=c_{ijlk}=c_{jikl}\,,\,, \tag{6}$$

By substituting Equations 1 to 6 into Equations 1 the resulting equations governing dynamic processes in *nonlocal* elasticity in the absence of body forces are then written as can be written as

$$\sigma_{ij,j}(\mathbf{u})=\left(1-\left(e_0a\right)^2\nabla^2\right)\rho\ddot{u}_i \tag{7}$$

3. NONLOCAL CHRISTOFFEL EQUATION AND ANALYSIS

In the nonlocal theory of elasticity, elasto-dynamical Equation 7 describing the inertial forces can be written with (the displacement) as

$$C_{ijkl}\frac{\partial^2 u_k}{\partial x_j\partial x_l}=\rho\left(1-\varepsilon^2\nabla^2\right)\frac{\partial^2 u_i}{\partial t^2}, \tag{8}$$

where $\varepsilon\left(=e_0a\right)$ is the nonlocality parameter defined in (1).

The displacement of plane wave can be described by any harmonic form as a function of time (e.g. Fedorov [32])

$$u_j=U_j\exp[i\xi(\mathbf{n.x}-ct)] \tag{9}$$

where ξ is the wave number, c is the phase velocity (= ω/ξ), ω is the circular frequency, U_j are the constants related to the amplitudes of displacement, n_k (k = 1, 2, 3) are the components of the unit vector $\mathbf{n}$ giving the direction of propagation. Inserting Equation 9 into Equations 8 generates the Christoffel equation of the form in nonlocal elasticity, we have

$$\Pi_{ik}U_k = \mathbf{0} \tag{10}$$

where

$$\Pi_{ik} = \Gamma_{ik} - \rho\left(1+\varepsilon^2\xi^2\right)c^2\delta_{ik} \tag{11}$$

Equations 10 can be rewritten in the matrix form as

$$\left[\boldsymbol{\Gamma} - V^2\mathbf{I}\right]\mathbf{U} = \mathbf{0} \tag{12}$$

Equation 12 is a general nonlocal Christoffel equation, where $\mathbf{I}$ is an identity matrix of order 3 and $V^2 = \rho\left(1+\varepsilon^2\xi^2\right)c^2$, in which δ_{ik} is the Kronecker delta, and Γ_{ik} are the Christofied stiffness as follows:

$$\Gamma_{ik} = \Gamma_{ki} = C_{ijkl}n_j n_l = \bar{\Gamma} \tag{13}$$

Here, n is the unit vector in the slowness direction; summation over repeated indices is implied. The Christoffel Equation 12 describes a standard eigenvalue V^2 eigenvector (**U**) problem for the matrix $\bar{\Gamma}$, with the eigenvalues are determined by

$$\det\left(\Pi_{ik}\right) = 0 \tag{14}$$

Therefore the eigenvalue solution is performed on the nonlocal Christoffel Equation 14. Specializing the above Equations 10-14 for a elastic solid of a cubic symmetry, which has three elastic constants C_{11}, C_{12} and C_{44}, consequently the local stress tensor σ_{kl} obeys

$$\sigma_{11} = C_{11}e_{11} + C_{12}(e_{22}+e_{33}),\ \sigma_{22} = C_{12}(e_{11}+e_{33}) + C_{11}e_{22}$$

$$\sigma_{33} = C_{12}(e_{11}+e_{22}) + C_{11}e_{33},\ \sigma_{23} = 2C_{44}e_{23},\ \sigma_{13} = 2C_{44}e_{13},\ \sigma_{12} = 2C_{44}e_{12} \tag{15}$$

Equation 15 represents the stress-strain relation for an elastic solid of a cubic symmetry,

The strain tensor e_{kl} for an elastic medium is defined as

$$e_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right). \tag{16}$$

Equation 16 are the strain tensor represents the relation in terms of displacement parameters.
Using (15), (16) in (13) we have

$$\Gamma_{11} = C_{11}n_1^2 + C_{44}(n_2^2 + n_3^2)$$

$$\Gamma_{12}=\left[(C_{12}+C_{44})n_1n_2\right]=\Gamma_{21}$$

$$\Gamma_{13}=(C_{12}+C_{44})n_1n_3=\Gamma_{31}$$

$$\Gamma_{22}=C_{11}n_2^2+C_{44}(n_1^2+n_3^2)$$

$$\Gamma_{23}=(C_{12}+C_{44})n_2n_3=\Gamma_{32}$$

$$\Gamma_{33}=C_{44}(n_1^2+n_2^2)+C_{11}n_3^2$$

Equation 12 leading to the cubic equations in $V^2=\left(\rho\left(1+\varepsilon^2\xi^2\right)c^2\right)$ for the wave velocity

$$V^3+F_1V^2+F_2V+F_3=0 \tag{17}$$

Where

$$F_1=-\left(C_{11}+2C_{44}\right),$$

$$F_2=C_{44}\left(2C_{11}+C_{44}\right)+\left(n_1^2n_2^2+n_3^2n_3^2+n_3^2n_1^2\right)\left(C_{12}+C_{44}\right)\left(C_{11}-C_{12}-2C_{44}\right),$$

$$F_3=C_{44}\left(C_{12}+C_{44}\right)^2\left(n_1^4+n_2^4+n_3^4-\left(n_1^6+n_2^6+n_3^6\right)\right)-\left(n_1^2n_2^2n_3^2\right)(2C_{44}-3C_{11}-2C_{12})\left(C_{12}+C_{44}\right)^2$$
$$-\left(C_{11}n_1^2+C_{44}\left(1-n_1^2\right)\right)\left(C_{12}n_2^2+C_{44}\left(1-n_2^2\right)\right)\left(C_{11}n_3^2+C_{44}\left(1-n_3^2\right)\right)$$

The roots of Equation 17 can be studied and represented graphically as three unique velocity surfaces in the nonlocal elasticity for any cubic material. It is convenient to examine and study the waves and slowness surfaces in [100], [110], and [111] planes of cubic crystals in nonlocal elasticity, intersection of these surfaces with the three principal orthogonal planes can be investigated in nonlocal elasticity. Relationships describing the wave velocity surfaces in the context of nonlocal elasticity for each plane can be derived using **(17).**

3.1. Propagation along a Cube Face

For propagation in the plane of a cube face (001), the above equations are simplified by considering $n_1=\cos(\phi), n_2=\sin(\phi),\ n_3=0$, where ϕ is the angle between the propagation direction $\mathbf{n}$ and the axis [100] of the crystal. Consequently, Equation 14 become

$$\begin{vmatrix} C_{11}n_1^2+C_{44}n_2^2-\rho\left(1+\varepsilon^2\xi^2\right)c^2 & \Gamma_{12} & 0 \\ \Gamma_{12} & C_{11}n_2^2+C_{44}n_1^2-\rho\left(1+\varepsilon^2\xi^2\right)c^2 & 0 \\ 0 & 0 & C_{44}(n_1^2+n_2^2)-\rho\left(1+\varepsilon^2\xi^2\right)c^2 \end{vmatrix}=0. \tag{18}$$

Equation 18 describes the Christoffel's nonlocal equation for the wave propagation along a cube face.
Simplifying (18) we obtain

$$\left(\rho\left(1+\varepsilon^2\xi^2\right)c^2-C_{44}\right)\left\{\begin{matrix}\left(\rho\left(1+\varepsilon^2\xi^2\right)c^2\right)^2 \\ -\left(C_{11}+C_{44}\right)\begin{pmatrix}\rho\left(1+\varepsilon^2\xi^2\right)c^2+C_{11}C_{44}\left(n_1^4+n_2^4\right) \\ +\left(C_{11}^2+C_{44}^2-C_{12}^2\right)\end{pmatrix}\end{matrix}\right\}=0. \tag{19}$$

Equation 19 demonstrate that a transverse wave polarized along OX_3, with velocity $c_t = \sqrt{C_{44}/\rho\left(1+\varepsilon^2\xi^2\right)}$, independent of the angle ϕ , influenced by the nonlocality parameter (ε) for any propagation direction in the (001) plane.

Solutions for the other two velocities for quasi-longitudinal and quasi-transverse waves are obtained by solving the second factor of Equation 19 we obtain

$$c_{ql,qt} = \left\{\left[\left(C_{44}+C_{11}\right) \pm \sqrt{\left(C_{11}-C_{44}\right)^2\left(n_1^2-n_2^2\right)+\left(C_{12}+C_{44}\right)^2 4n_1^2n_2^2}\,\right]\Big/2\rho\left(1+\varepsilon^2\xi^2\right)c^2\right\}^{\frac{1}{2}}. \tag{20}$$

These curves exhibit the maximal symmetry of the cubic system - the stiffness tensor has the same form for all cubic classes, and so is invariant for the symmetry operations in the holosymmetric class. The velocity of the quasi-transverse wave has extrema in the [100] and [110] directions, given by

$$c_{qt}[100] = \sqrt{\frac{C_{44}}{\rho\left(1+\varepsilon^2\xi^2\right)}}, c_{qt}[110] = \sqrt{\frac{C_{11}-C_{12}}{2\rho\left(1+\varepsilon^2\xi^2\right)}}. \tag{21}$$

Clearly these have dependence on the non-locality parameter. When $\phi = 0$ or $\phi = \pi/2$ refers to a pure longitudinal transverse wave in nonlocal elasticity. In all other directions, equation (20) remains coupled and give velocities for quasi-longitudinal and quasi-transverse waves.

3.2. Propagation in a Diagonal Plane

In order to study the wave propagation in the (111) plane, it is convenient to transform to a new set of axes X_1', X_2' and X_3' such that the X_1' axis coincides with the (111) direction of the cubic diagonal; then the $X_2'X_3'$ plane would represent the (111) symmetry plane. On considering the diagonal plane $(1\bar{1}0)$, the suitable axes $OX_1'X_2'X_3'$ can be obtained by rotation through $\frac{\pi}{4}$ about OX_3 from the $OX_1X_2X_3$. These enable the velocities to be expressed in terms of C_{11}, C_{12} and C_{44}. The velocity of the transverse wave is

$$c_t = \sqrt{\frac{1}{\rho\left(1+\varepsilon^2\xi^2\right)}\left[C_{44}\cos^2(\theta)+\left(C_{11}-C_{12}\right)\sin^2(\theta)/2\right]} \tag{22}$$

and the velocities of the quasi-longitudinal and quasi-transverse waves, respectively c_{ql} and c_{qt} are given by

$$c_{ql,qt} = \sqrt{\frac{1}{2\rho\left(1+\varepsilon^2\xi^2\right)}\left\{\begin{array}{l} C_{44}+\left(\dfrac{2C_{44}+C_{11}+C_{12}}{2}\right)\sin^2(\theta)+C_{11}\cos^2(\theta) \\ \pm\sqrt{\left[\dfrac{C_{11}+C_{12}}{2}\sin^2(\theta)+\left(C_{44}-C_{11}\right)\cos^2(\theta)\right]+\left(C_{12}+C_{44}\right)^2\sin^2(2\theta)} \end{array}\right\}}. \tag{23}$$

Clearly c_t, Equation 22 , and c_{ql} and c_{qt} in Equation 23 are influenced by the nonlocality parameter (ε) for any propagation in a diagonal plane.

Equation 23 shows that c_t corresponds to a pure transverse wave. In nonlocal elasticity, become dispersive but non-attenuating and are influenced by non-locality parameter, while theses waves are non-dispersive and non-attenuating in its counterpart local theory of elasticity. It can be seen that the transverse wave in nonlocal elastic solid travels slower than that of longitudinal wave even in the presence non-locality parameter likewise as in case of classical continuum mechanics.

3.3. Wave Surface along a Cube Edge

If we consider the problem of progressive waves propagation along the [001] edge of a cubic crystal, we take $n_1 = n_2 = 0$ and $n_3 = 1$. In this case the dispersion Equation 17 has the form:

$$\left(C_{44} - \rho\left(1+\varepsilon^2\xi^2\right)c^2\right)\left(C_{44} - \rho\left(1+\varepsilon^2\xi^2\right)c^2\right)\left(C_{11} - \rho\left(1+\varepsilon^2\xi^2\right)c^2\right) = 0. \qquad (24)$$

Equation 24 demonstrate that in the case of waves surface along a cube edge waves polarized are in the plane (001) on mutually perpendicular directions, one wave corresponds to pure longitudinal and two waves corresponds to a pure transverse waves. Clearly all the waves are dispersive and are influenced by the nonlocality parameter

3.4. Wave Surface along a Cube Face Edge

On considering the problem of plane waves propagation along the (001) plane of a cubic crystal the problem of waves propagation along the [001] edge of a cubic crystal, we take $n_1 = \cos(\phi), n_2 = \sin(\phi)$ and $n_3 = 0$, where ϕ is the angle between the propagation direction n and the [100] axis of the crystal. In this case the dispersion Equation 21 has the form:

$$\left(C_{44} - \rho\left(1+\varepsilon^2\xi^2\right)c^2\right)\left\{\left(\rho\left(1+\varepsilon^2\xi^2\right)c^2\right)^2 + A\left(\rho\left(1+\varepsilon^2\xi^2\right)c^2\right) + B\right\} = 0. \qquad (25)$$

where

$$A = -\left(C_{11} + C_{44}\right), B = C_{11}C_{44} - F_1F_2\cos^2(\phi)\sin^2(\phi),$$

$$F_1 = \left(C_{11} + C_{12}\right)\ F_2 = C_{12} + 2C_{44} - C_{11}.$$

As regards the polarization of the waves, Equation 25 demonstrates that in this case one wave corresponds to corresponds to a pure *transverse wave*. The remaining two waves are polarized in the plane (001) on mutually perpendicular directions, one being quasi-transverse, and the other quasi-longitudinal. It is interesting to note that the directions of polarization of the last two waves are influenced by the nonlocality parameter.

3.5. Special Case

Cubic symmetry with $n_1 = n_2 = 1/\sqrt{2}$. Equation 19 degenerates into one longitudinal, and two transverse waves $c_1 = \sqrt{C_{44}/\rho\left(1+\varepsilon^2\xi^2\right)}$,

$c_2=\sqrt{\left(C_{11}-C_{12}\right)/2\rho\left(1+\varepsilon^2\xi^2\right)}$ and

$c_3=\sqrt{\left(C_{11}+2C_{44}+C_{12}\right)/2\rho\left(1+\varepsilon^2\xi^2\right)}$. All these waves are dispersive and depends on the nonlocality parameter.

Where the nonlocality parameter $\varepsilon\left(=e_0a\right)=0$, then Equation 19 reduces to $c_t=\sqrt{C_{44})/\rho}$,

$$c_{ql,qt}=\left\{\left[C_{44}+C_{11}\pm\sqrt{\left(C_{66}+C_{11}\right)^2-4A}\right]\Big/2\rho c^2\right\}^{\frac{1}{2}}, \tag{26}$$

which are wave velocities of longitudinal and transverse waves in the classical continuum mechanics theory, (local theory) which becomes non-dispersive non-attenuating.

4. PHASE AND GROUP VELOCITIES

The phase vectors depict the direction of the phase velocity whereas group vectors depict the direction of the group velocity. Based on the definition of phase velocity $c_p=\omega/\xi$, we can replace $\omega=\xi.c_p$, then we obtain group velocity as

$$c_g=\frac{\partial\omega}{\partial\xi}=\frac{\partial\left(\zeta.c_p\right)}{\partial\xi}=c_p+\xi\frac{\partial\left(c_p\right)}{\partial\xi}. \tag{27}$$

Equation 27 represents the relation phase velocity whereas group velocity.

From Equation 26 we observed that c_t remain in pure modes and c_{ql}, c_{qt} (longitudinal and transverse waves) become dispersive in nonlocal elasticity and are influenced by non-locality parameter. Waves are propagating in a dispersive medium; they will have different velocities and thus the superposed wave will have a phase velocity c_p that is different from its group velocity c_g. If, $c_p>c_g$, exhibiting that dispersion is normal. In case of classical (local) continuum mechanics when the non local parameter $\varepsilon\left(=e_0a\right)=0$, then $c_p=c_g$ (m/s) (longitudinal wave) and $c_p=c_g$ (m/s) (transverse wave) for cubic materials. Thus waves are non-dispersive in its counterpart local theory of elasticity.

5. SLOWNESS SURFACE

For $\phi=0$ (i.e. [100] axis), cubic symmetry with $n_1=1, n_2=0,$ from relations (21) yields the wave

$$\left[\left(C_{44}-\rho\left(1+\varepsilon^2\xi^2\right)c^2\right)\right]^2\left(C_{11}-\rho\left(1+\varepsilon^2\xi^2\right)c^2\right)=0. \tag{28}$$

In this case waves are polarized in the plane (001) on mutually perpendicular directions, from Equation 28, it is observed that one wave corresponds to pure longitudinal and two waves corresponds to a pure transverse waves. Clearly all the waves are dispersive and are influenced by the nonlocality parameter (ε).

For $\phi=\dfrac{\pi}{2}$ (i.e. [010] axis), cubic symmetry with $n_1=0, n_2=1,$ from relations (21) we obtain the following wave velocities:

$$\left[\left(C_{44}-\rho\left(1+\varepsilon^2\xi^2\right)c^2\right)\right]^2\left(C_{11}-\rho\left(1+\varepsilon^2\xi^2\right)c^2\right)=0. \tag{29}$$

In this case, from Equation 29, it is observed that one corresponding to a pure longitudinal wave, and the others to a pure transverse waves and are influenced by the nonlocality parameter (ε)..

For $\phi=\dfrac{\pi}{4}$ (i.e. [110] axis) cubic symmetry with $n_1=n_2=1/\sqrt{2}$. Equation 21 degenerates into one longitudinal, and two transverse waves $c_1=\sqrt{C_{44}/\rho\left(1+\varepsilon^2\xi^2\right)}$,

$c_2=\sqrt{\left(C_{11}-C_{12}\right)/2\rho\left(1+\varepsilon^2\xi^2\right)}$ and

$c_3=\sqrt{\left(C_{11}+2C_{44}+C_{12}\right)/2\rho\left(1+\varepsilon^2\xi^2\right)}$. All these waves are dispersive and depends on the nonlocality parameter.

Where the nonlocality parameter $\varepsilon\left(=e_0a\right)=0$, then Equation 23 reduces to $c_t=\sqrt{C_{44})/\rho}$,

$$c_{ql,qt}=\left\{\left[C_{44}+C_{11}\pm\sqrt{\left(C_{44}+C_{11}\right)^2-4A}\right]\Big/2\rho c^2\right\}^{\frac{1}{2}}, \tag{30}$$

Equation 30 exhibit that wave velocities of longitudinal and transverse waves in the classical continuum mechanics theory, (local theory) which becomes non-dispersive non-attenuating. Where the nonlocality parameter $\varepsilon\left(=e_0a\right)=0$, results in (25)-(27) reduce to the corresponding local elasticity.

Equation 17 is a cubic characteristic polynomial equation in V^2 in nonlocal elasticity, and hence has three eigenvalues corresponding to a longitudinal wave, and two transverse waves in the same manner as in local case. The largest eigenvalue of Equation 17 corresponds to the longitudinal wave propagation an is uniquely defined, because the velocity of the longitudinal wave is always greater than those of the transverse. Therefore the slowness sheet is the innermost one and is away from the other two which are coincident (in isotropic case) is a function of non local parameter. Further the polarization vector of the longitudinal wave is tangent to the wave front normal and the polarization vectors of the transverse waves are normal vector of the longitudinal wave, with the three eigenvectors forming an orthogonal system.

6. NUMERICAL DISCUSSION

The numerical computation is carried out over cubic materials. Physical data of the substances that crystallize in the cubic system has only three independent stiffness constants are given in Table 1.

From the Equation 21 the velocity of the quasi-transverse wave has extrema in the [100] and [110] directions, in which propagation direction [100]; Polarization[100] (Longitudinal) and (100) plane (Transverse) velocity is given by in Table 2. From the table values for crystals Silicon(Si), Aluminum (Al), Copper (Cu), Nickel (Ni), Gold (Au) found that velocities of longitudinal and transverse waves continuously decreases with increases of non-locality parameter when it is greater than 0.001 and the no change is observed when non local parameter is less than 0.001.

In Table 3, propagation direction [110]; polarization[100] (longitudinal) and [110] plane (transverse) and and [001] (transverse) velocity: $C_L = \sqrt{\dfrac{C_{11}+C_{12}+2C_{44}}{2\rho\left(1+\varepsilon^2\xi^2\right)}}$ (longitudinal), $C_{T1} = \sqrt{\dfrac{C_{11}-C_{12}}{2\rho\left(1+\varepsilon^2\xi^2\right)}}$ $C_T = \sqrt{\dfrac{C_{44}}{\rho\left(1+\varepsilon^2\xi^2\right)}}$ (Transverse) with nonlocality parameter (ε) for the **Cubic Crystals in** Table 1 **is** tabulated. It is observed that wave velocity pattern of C_T **(Transverse)** exhibits no change as in the previous case, whereas C_L remain slight faster in this case for all values of the non local parameter from the previous case, at the same time C_{T1} remain slower than both the C_T and C_L . Velocities of longitudinal C_L and transverse (C_L ,C_{T1}) waves continuously decreases with increases of non-locality parameter, when it is greater than 0.001 and the no change is observed when non local parameter is less than 0.001. In this case relation among the three velocities is $C_{T1} < C_T < C_L$ in nonlocal elasticity.

In Table 4, propagation direction [111]; polarization[111] (longitudinal) and (111) plane (transverse) velocity: $C_L = \sqrt{\dfrac{C_{11}+2C_{12}+2C_{44}}{3\rho\left(1+\varepsilon^2\xi^2\right)}}$ (longitudinal), $C_T = \sqrt{\dfrac{C_{11}-C_{12}+C_{44}}{3\rho\left(1+\varepsilon^2\xi^2\right)}}$ with nonlocality parameter (ε) for the **cubic** crystals in Table 1 is tabulated. In this case, similar behavior is observed as in the previous case for all values of non local parameter (which were taken Table 2, Table 3). In this direction longitudinal wave speed is higher than the previous values, and exhibit the same behavior but with different speeds.

Using the data in the Table 1 the velocity of the quasi-transverse wave has extrema in the [100] and [110] directions, depends upon nonlocality parameter (ε) given by

$C_T[100] = \sqrt{\dfrac{C_{44}}{\rho\left(1+\varepsilon^2\xi^2\right)}}$, $C_T[110] = \sqrt{\dfrac{C_{11}-C_{12}}{2\rho\left(1+\varepsilon^2\xi^2\right)}}$, $\dfrac{C_T[100]}{C_T[110]} = \sqrt{\dfrac{2C_{44}}{C_{11}-C_{12}}} = \sqrt{A_F}$ The ratio of these values is independent of non local parameter (ε)

Here A_F is the anisotropy factor for crystals of a cubic symmetry, which is clearly not influenced by the non local parameter. Table 5 tabulate the anisotropy factor of crystals in Table 1 of a cubic symmetry.

Figures 1 to Figure 3 exhibit the wave surfaces of quasi-longitudinal, quasi-transverse and transverse for silicon for the nonlocal parameter $\varepsilon = 0,\ 0.01,$ and 0.5.

Figures 4 to Figure 6 display the slowness surfaces of quasi-longitudinal, quasi-transverse and transverse for silicon for nonlocal parameter $\varepsilon = 0,\ 0.01,$ and 0.1.

Variation of phase velocity speed with nonlocality parameter for Gold (Au), Silicon(Si), Aluminum (Al), Copper (Cu) and Nickel (Ni) with **angle** $\phi = \pi/4$ is plotted in Figures 7 to Figure 11 respectively. These figures show that there is steep gradient in the neighborhood of $\varepsilon = 0.01$ in all of these curves before tends to become horizontal.

Further variation of phase wave speed with nonlocality parameter for Nickel (Ni) and Silicon(Si), with angle $\phi = \pi/2$ and 0^0 is exhibited in Figures 12 to Figure 13 respectively. In Figure 14, phase and group velocities variation with nonlocality parameter for quasi-longitudinal, quasi-transverse and transverse for Silicon(Si) when angle ϕ between the propagation direction is $\pi/4$ is plotted.

Slowness surfaces for all the three modes in the absence of nonlocality parameter for Silicon(Si), Cupper (Cu), Nickle (Ni) and Gold (Au) when propagation direction is in a diagonal plane is plotted in Figures 15 to Figure 18 .

Table -1. Physical constants of cubic crystals in $10^{10}\left(\text{N/m}^2\right)$ or $10^{11}\left(\text{dynes/cm}^2\right)$ and density is in $\left(\text{g/cm}^3\right)$

Cubic Crystals Material	C_{11} $10^{10}\left(\text{N/m}^2\right)$	C_{12} $10^{10}\left(\text{N/m}^2\right)$	C_{44} $10^{10}\left(\text{N/m}^2\right)$	ρ $\left(\text{g/cm}^3\right)$
Silicon(Si)	16.56	6.39	7.59	2.329
Aluminum (Al)	10.73	6.08	2.83	2.709
Copper (Cu)	17.0	12.0	7.55	8.93
Nickel (Ni)	25.3	15.5	12.4	8.90
Gold (Au)	19.25	16.30	4.24	19.3

Source: Physical constants of cubic crystals Fedorov [32].

Table-2. Propagation direction [100]; Polarization[100] (Longitudinal) and (100) plane (Transverse) Velocity: $C_L = \sqrt{\dfrac{C_{11}}{\rho\left(1+\varepsilon^2\xi^2\right)}}$ (Longitudinal), $C_T = \sqrt{\dfrac{C_{44}}{\rho\left(1+\varepsilon^2\xi^2\right)}}$ (Transverse) with nonlocality parameter (ε) for the Cubic Crystals in Table 1.

Nonlocality parameter		Velocity				
		Si (m/s)	Al (m/s)	Cu (m/s)	Ni (m/s)	Au (m/s)
$\varepsilon \le 1.10^{-3}$	C_L	8432.29	6293.55	4663.14	5331.69	3158.18
	C_T	5842.5	3232.13	2907.69	3732.64	1482.19
$\varepsilon = 0.01$	C_L	8431.87	6293.23	4362.92	5331.43	3158.02
	C_T	5842.21	3231.97	2907.54	3732.45	1482.12
$\varepsilon = 0.1$	C_L	8390.44	6262.31	4341.48	5305.23	3142,51
	C_T	5813.51	3216.09	2893.26	3714.11	1474.84
$\varepsilon = 0.5$	C_L	7542.07	5629.12	3902.51	4768.81	2824.76
	C_T	5225.69	2890.9	2600.71	3338.57	1325.71
$\varepsilon = 1$	C_L	5962.53	4450.21	3085.2	3770.08	3770.08
	C_T	4131.27	2285.46	2056.05	2639.37	**1048.07**
$\varepsilon = 1.5$	C_L	4677.39	3491.03	2420.23	2957.49	1751.84
	C_T	3420.84	1792.86	1612.89	2070.49	822.17

Table-3. The velocity of the quasi-transverse wave has extrema in the [100] and [110] directions, given by propagation direction [110]; polarization[100] (longitudinal) and [110] plane (transverse) and [001] (transverse) velocity: $C_L=\sqrt{\dfrac{C_{11}+C_{12}+2C_{44}}{2\rho\left(1+\varepsilon^2\xi^2\right)}}$ (longitudinal), $C_{T1}=\sqrt{\dfrac{C_{11}-C_{12}}{2\rho\left(1+\varepsilon^2\xi^2\right)}}$ $C_T=\sqrt{\dfrac{C_{44}}{\rho\left(1+\varepsilon^2\xi^2\right)}}$ (transverse) with nonlocality parameter (ε) for the cubic crystals in Table 1.

Nonlocality parameter		Velocity				
		Si (m/s)	Al (m/s)	Cu (m/s)	Ni (m/s)	Au (m/s)
$\varepsilon \le 1.10^{-3}$	C_L	9132.62	6439.94	4969.11	6070.74	3377.39
	C_T	5842.50	3232.13	2907.69	3732.64	1482.19
	C_{T1}	4672.62	2929.59	1673.19	2364.40	874.21
$\varepsilon = 0.01$	C_L	9132.17	6439.62	4968.86	6070.44	3377.22
	C_T	5842.21	3231.97	2907.54	3732.45	1482.12
	C_{T1}	4672.39	2929.44	1673.10	2346.29	874.17
$\varepsilon = 0.1$	C_L	9087.30	6407.98	4944.45	6040.62	3360.62
	C_T	5813.51	3216.09	2893.26	3714.11	1474.84
	C_{T1}	4649.43	2915.05	1664.88	2334.76	869.87
$\varepsilon = 0.5$	C_L	8168.47	5760.06	4444.51	5429.84	3020.83
	C_T	5225.69	2890.90	2600.71	3338.57	1325.71
	C_{T1}	4179.32	2620.31	1496.54	2098.69	781.92
$\varepsilon = 1$	C_L	6457.74	4553.73	3513.69	4292.66	2388.17
	C_T	4131.27	2285.46	2056.05	2639.37	1048.07
	C_{T1}	3304.04	2071.53	1183.12	1659.16	618.16
$\varepsilon = 1.5$	C_L	5065.87	3572.24	2756.37	3367.44	1873.44
	C_T	3420.84	1792.86	1612.89	2070.49	822.17
	C_{T1}	2691.90	1625.04	928.12	1301.55	484.93

Table-4. Propagation direction [111]; Polarization [111] (Longitudinal) and (111) plane (Transverse) Velocity: $C_L=\sqrt{\dfrac{C_{11}+2C_{12}+2C_{44}}{3\rho\left(1+\varepsilon^2\xi^2\right)}}$ (Longitudinal), $C_T=\sqrt{\dfrac{C_{11}-C_{12}+C_{44}}{3\rho\left(1+\varepsilon^2\xi^2\right)}}$ with nonlocality parameter (ε) for the Cubic Crystals in Table 1.

Nonlocality parameter		Velocity				
		Si (m/s)	Al (m/s)	Cu (m/s)	Ni (m/s)	Au (m/s)
$\varepsilon \le 1.10^{-3}$	C_L	9354.43	6488.01	5155.30	6297.85	3227.95
	C_T	5092.53	3033.79	2164.39	2883.51	1114.36
$\varepsilon = 0.01$	C_L	9353.96	6487.68	5155.04	6297.54	3227.79
	C_T	5092.28	3033.64	2164.28	2883.36	1114.30
$\varepsilon = 0.1$	C_L	9308.01	6455.81	5129.71	6266.60	3211.93

	C_T	5067.26	3018.73	2163.65	2869.20	1108.83
$\varepsilon = 0.5$	C_L	8366.86	5803.05	4611.04	5632.97	2887.17
	C_T	4554.90	2713.51	1935.89	2579.09	996.71
$\varepsilon = 1$	C_L	6614.58	4587.71	3645.35	4453.25	2282.51
	C_T	3600.96	2145.21	1530.45	2038.95	787.97
$\varepsilon = 1.5$	C_L	5188.90	3598.90	2859.64	3493.42	1790.55
	C_T	2824.83	1682.84	1200.59	1599.48	618.14

Table-5. Anisotropy Factor (A_F).

Cubic Crystals Material	Anisotropy Factor
Silicon(Si)	1.563
Aluminum (Al)	1.217
Copper (Cu)	3.02
Nickel (Ni)	2.531
Gold (Au)	2.875

Source: Physical constants of cubic crystals Fedorov [32].

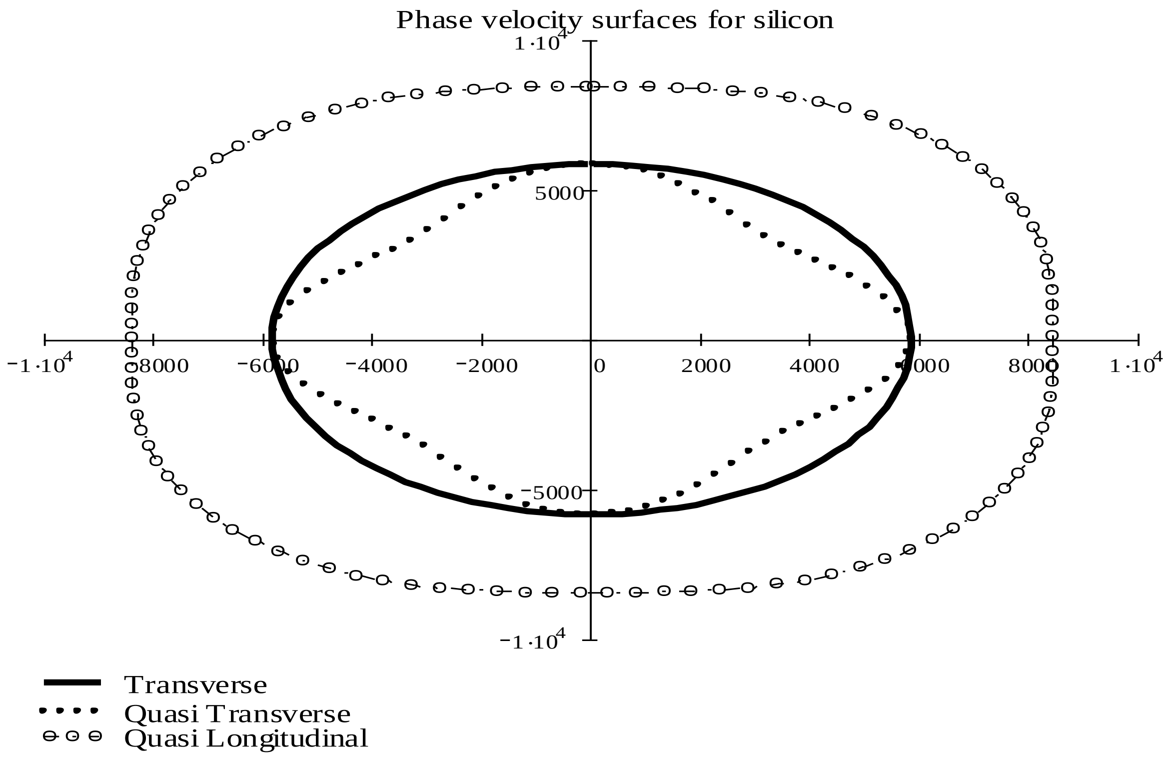

Figure-1. Wave surfaces of quasi-longitudinal, quasi-transverse and transverse for silicon for in the absence of nonlocal parameter.

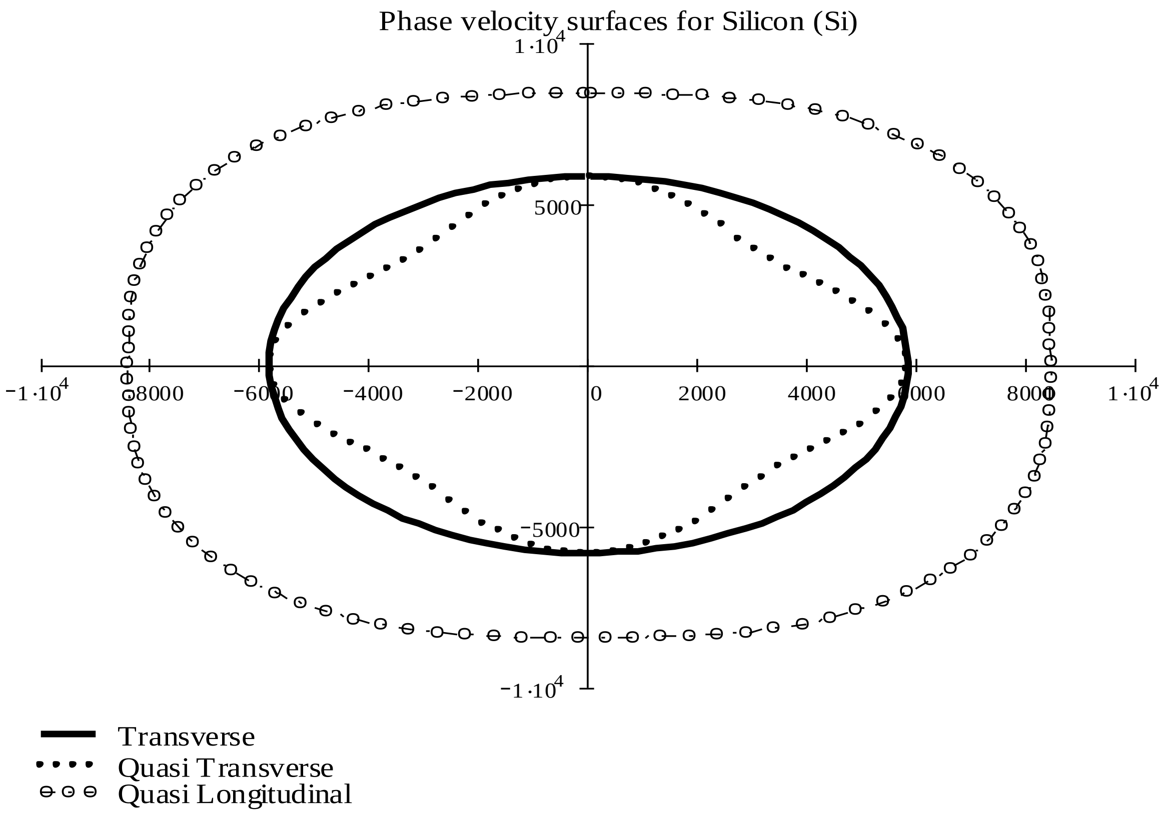

Figure-2. Wave surfaces of quasi-longitudinal, quasi-transverse and transverse for silicon for nonlocal parameter $\varepsilon = 0.01$.

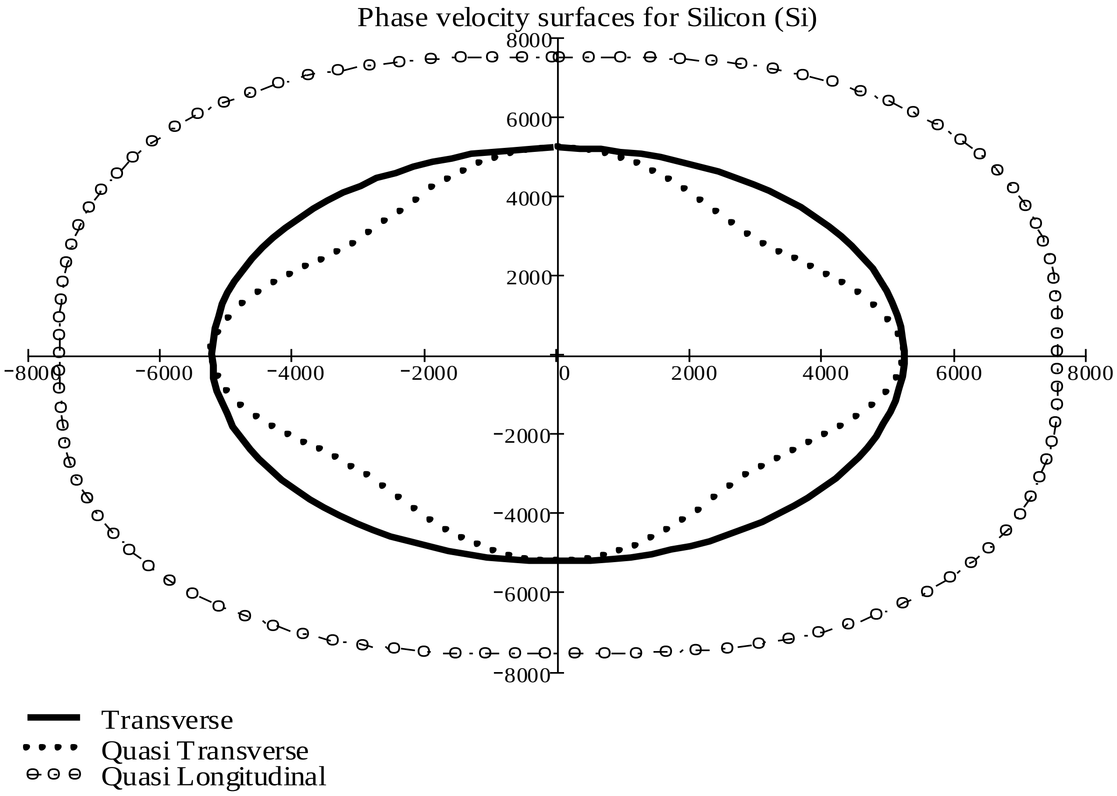

Figure-3. Wave surfaces of quasi-longitudinal, quasi-transverse and transverse for silicon for nonlocal parameter $\varepsilon = 0.5$.

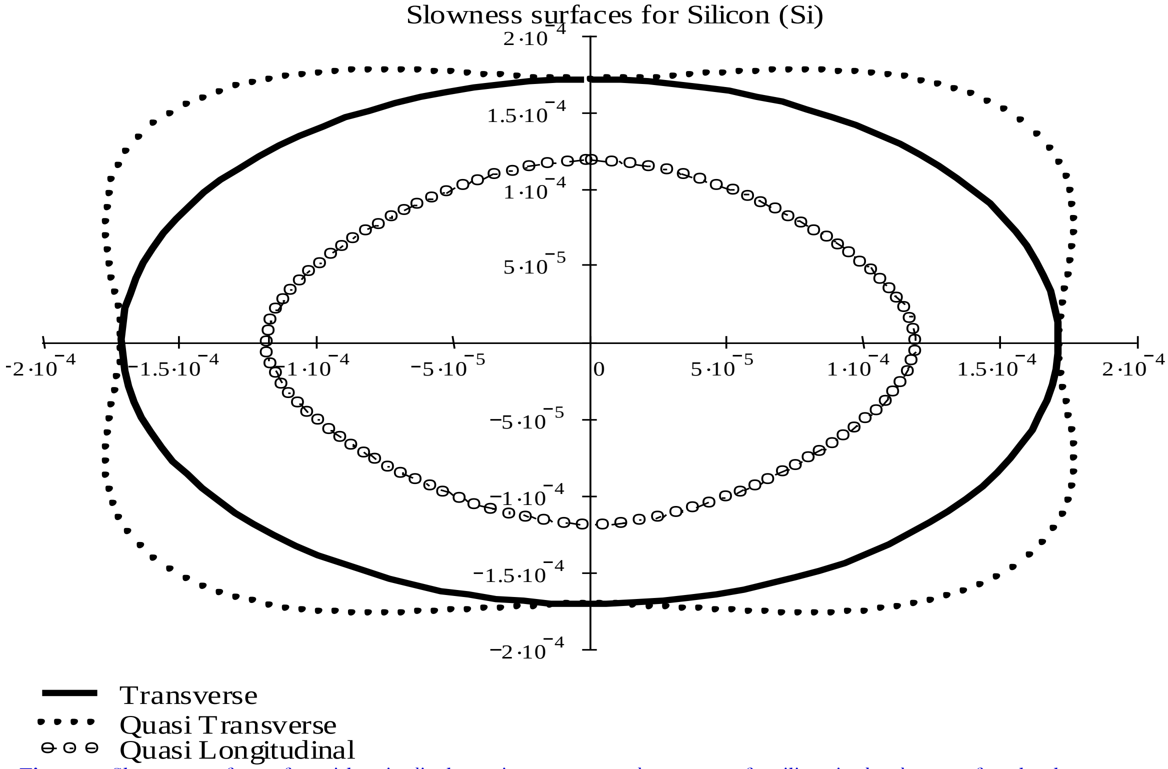

Figure-4. Slowness surfaces of quasi-longitudinal, quasi-transverse and transverse for silicon in the absence of nonlocal parameter.

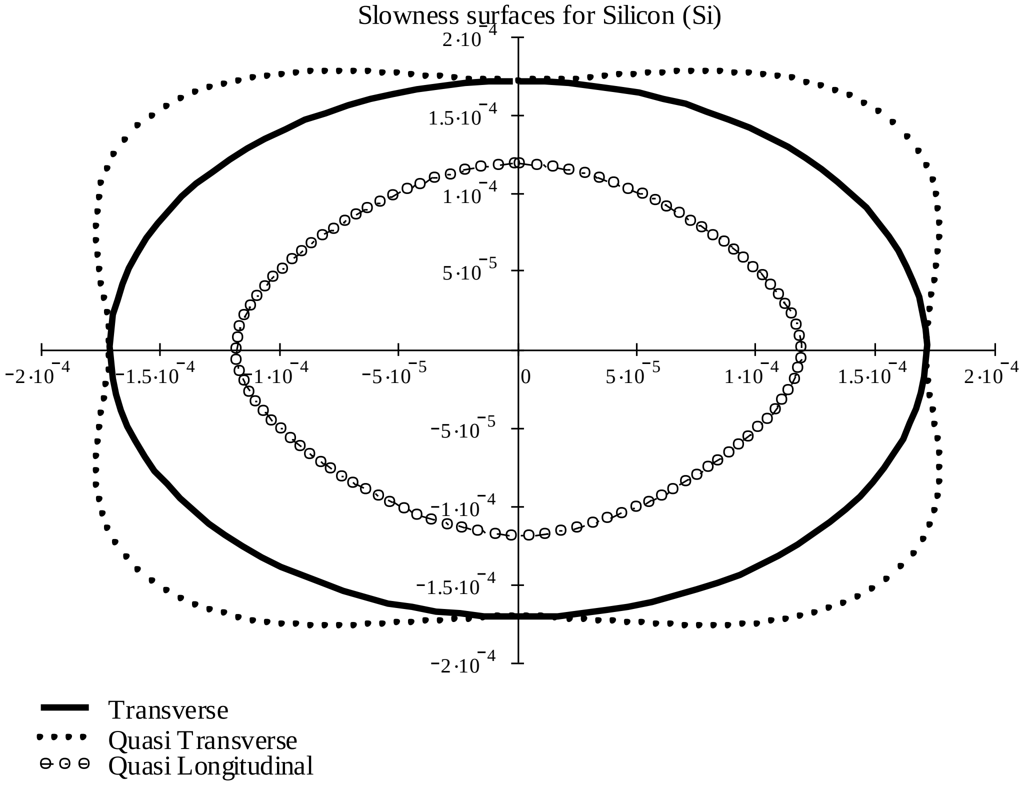

Figure-5. Slowness surfaces of quasi-longitudinal, quasi-transverse and transverse for silicon for nonlocal parameter $\varepsilon = 0.01$.

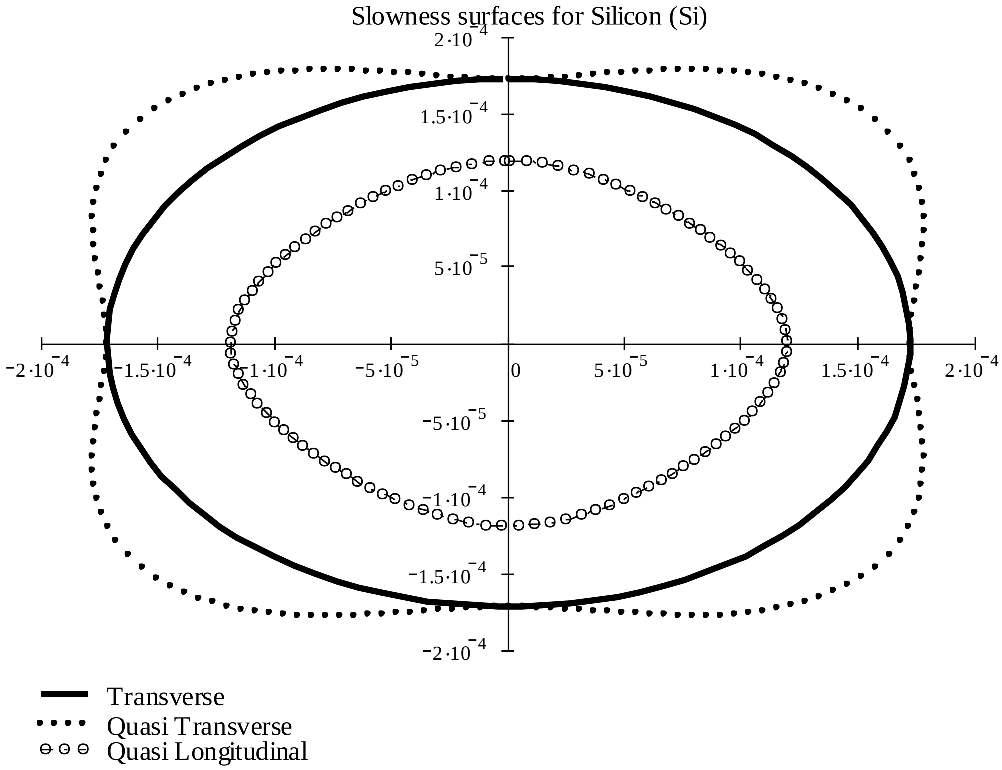

Figure-6. Slowness surfaces of quasi-longitudinal, quasi-transverse and transverse for silicon for nonlocal parameter $\varepsilon = 0.1$.

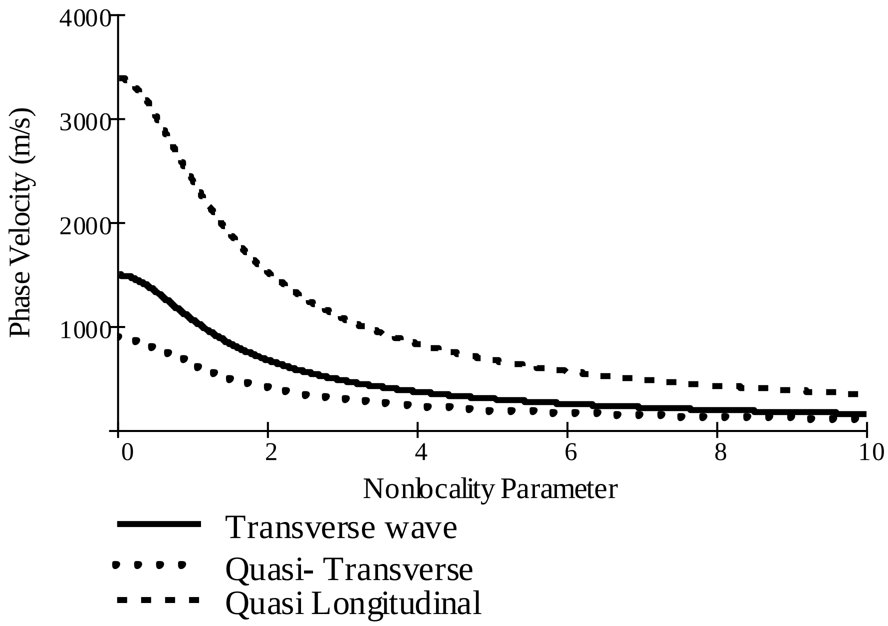

Figure-7. Variation of phase velocity with nonlocality parameter for Gold (Au) when angle ϕ between the propagation direction is $\pi/4$.

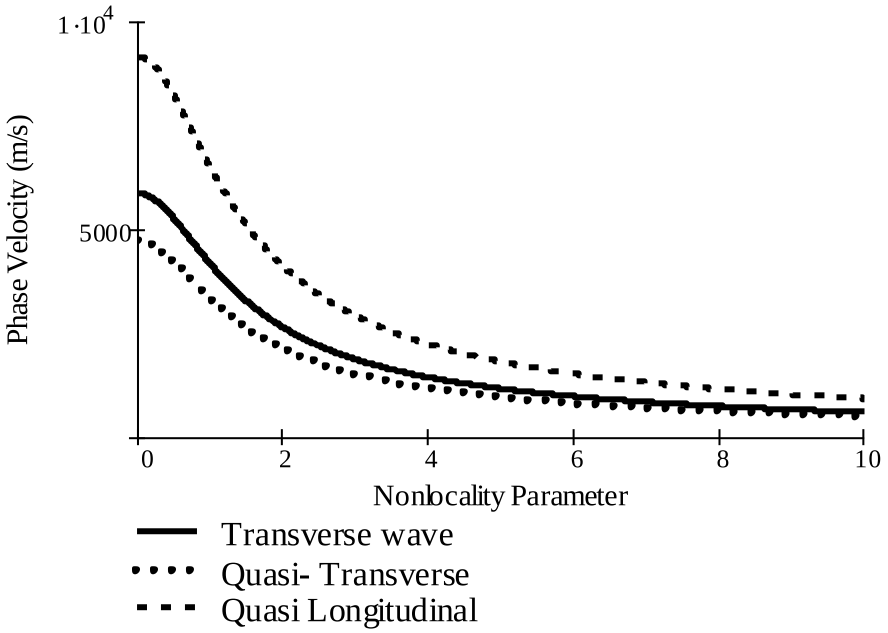

Figure-8. Variation of phase velocity with nonlocality parameter for Silicon(Si) when angle ϕ between the propagation direction is $\pi/4$.

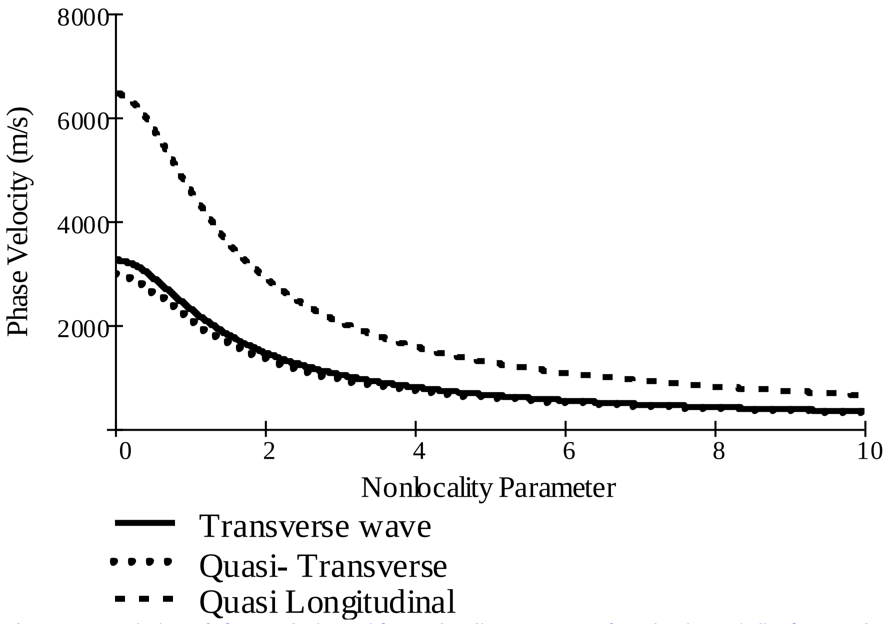

Figure-9. Variation of phase velocity with nonlocality parameter for Aluminum (Al) when angle ϕ between the propagation direction **is** $\pi/4$.

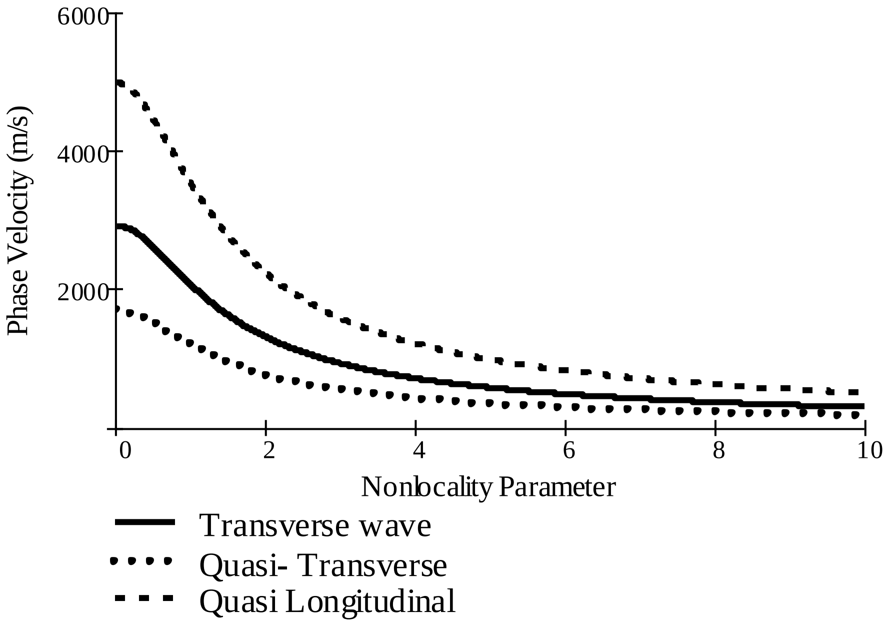

Figure-10. Variation of phase velocity with nonlocality parameter for Copper (Cu) when angle ϕ between the propagation direction is $\pi/4$.

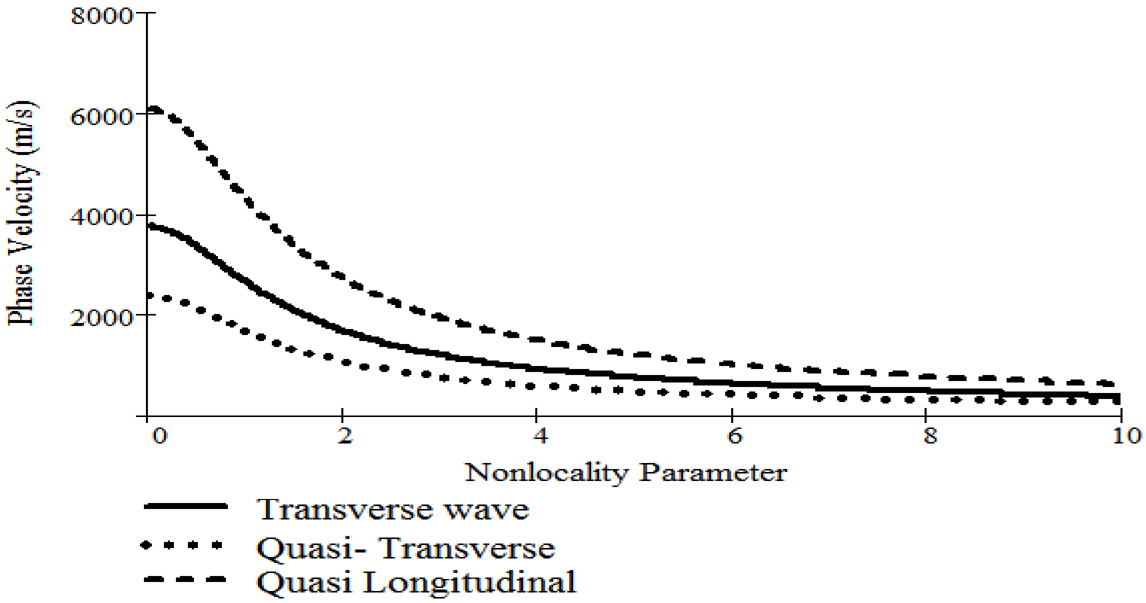

Figure-11. Variation of phase velocity with nonlocality parameter for Nickel (Ni) when angle ϕ between the propagation direction is $\pi/4$.

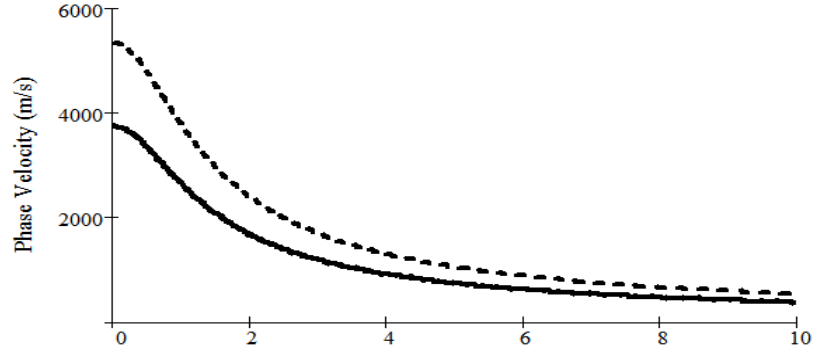

Figure-12. Variation of phase velocity with nonlocality parameter for Nickel (Ni) when angle ϕ between the propagation direction **is** $\pi/2$.

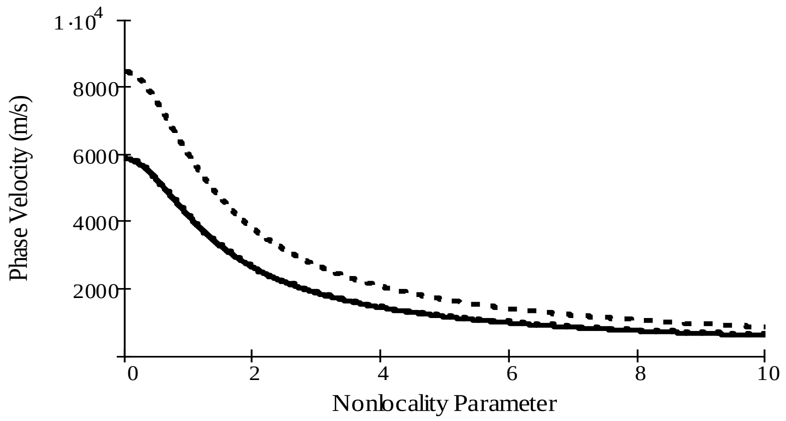

Figure-13. Variation of phase velocity with nonlocality parameter for Silicon(Si) when angle ϕ between the propagation direction is 0^0.

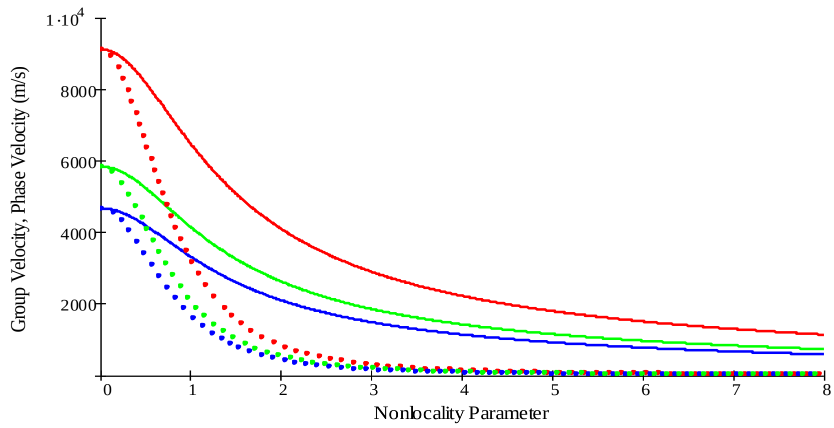

Figure-14. Variation of phase and group velocities with nonlocality parameter for Silicon(Si) when angle ϕ between the propagation direction **is** $\pi/4$ for all the three modes.

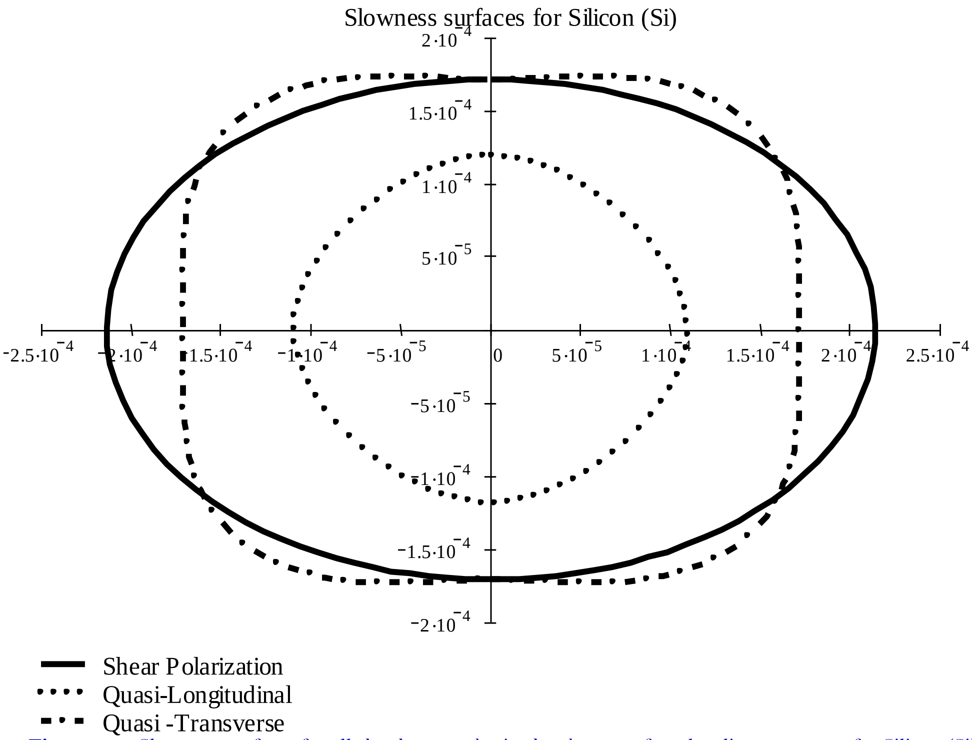

Figure-15. Slowness surfaces for all the three modes in the absence of nonlocality parameter for Silicon (Si) when propagation direction is in Propagation in a diagonal plane.

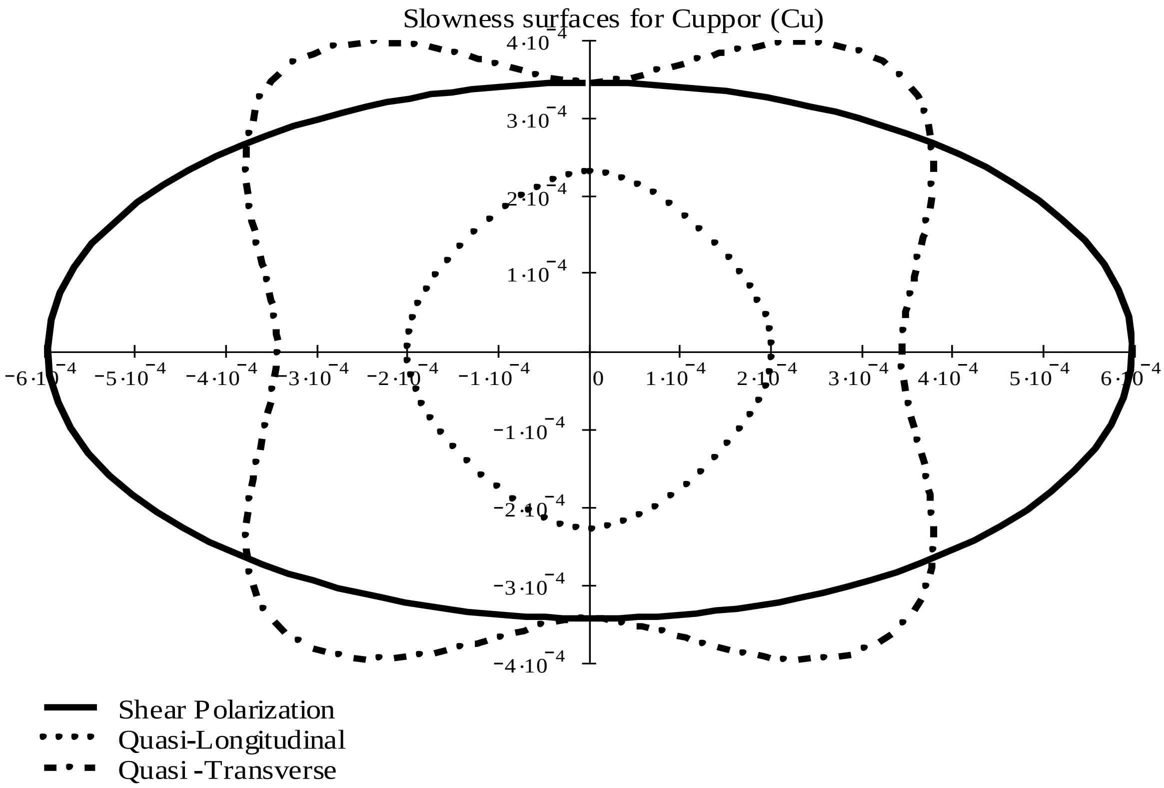

Figure-16. Slowness surfaces for all the three modes in the absence of nonlocality parameter for Cupper (Cu) when propagation direction is in Propagation in a diagonal plane.

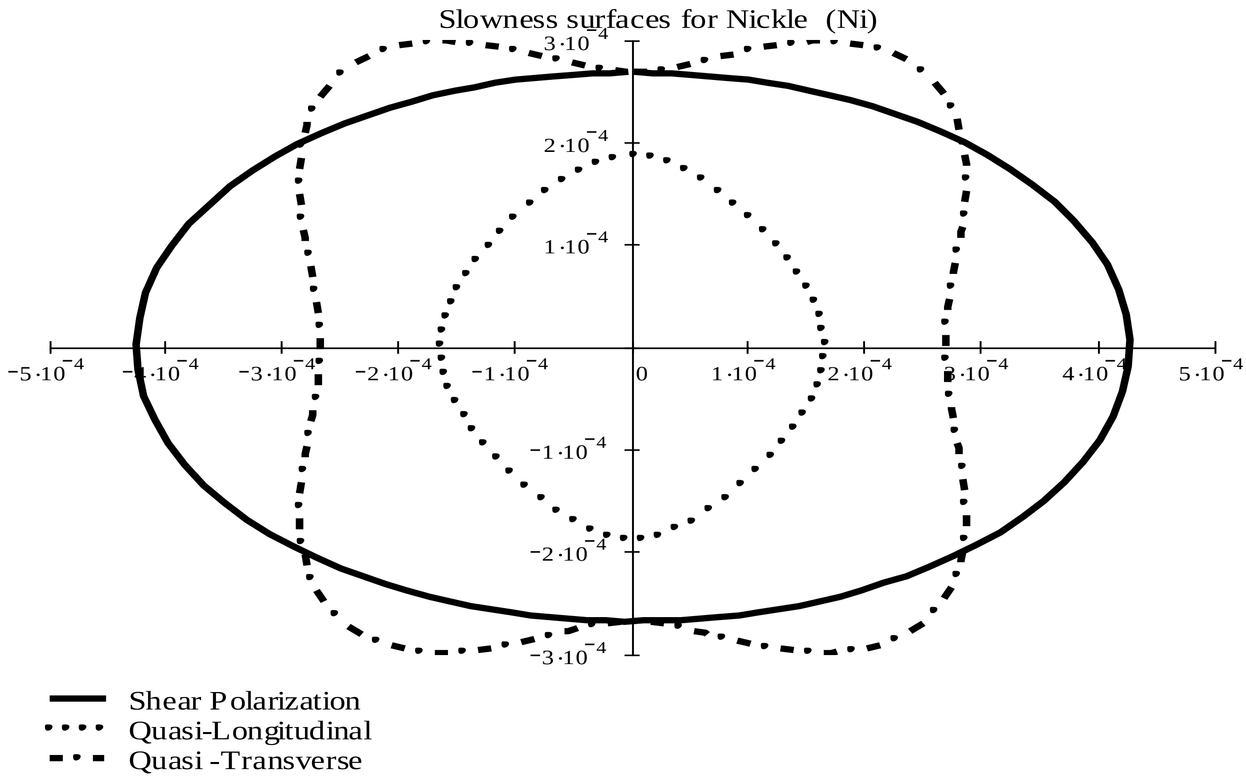

Figure-17. Slowness surfaces for all the three modes in the absence of nonlocality parameter for Nickle (Ni) when propagation direction is in Propagation in a diagonal plane.

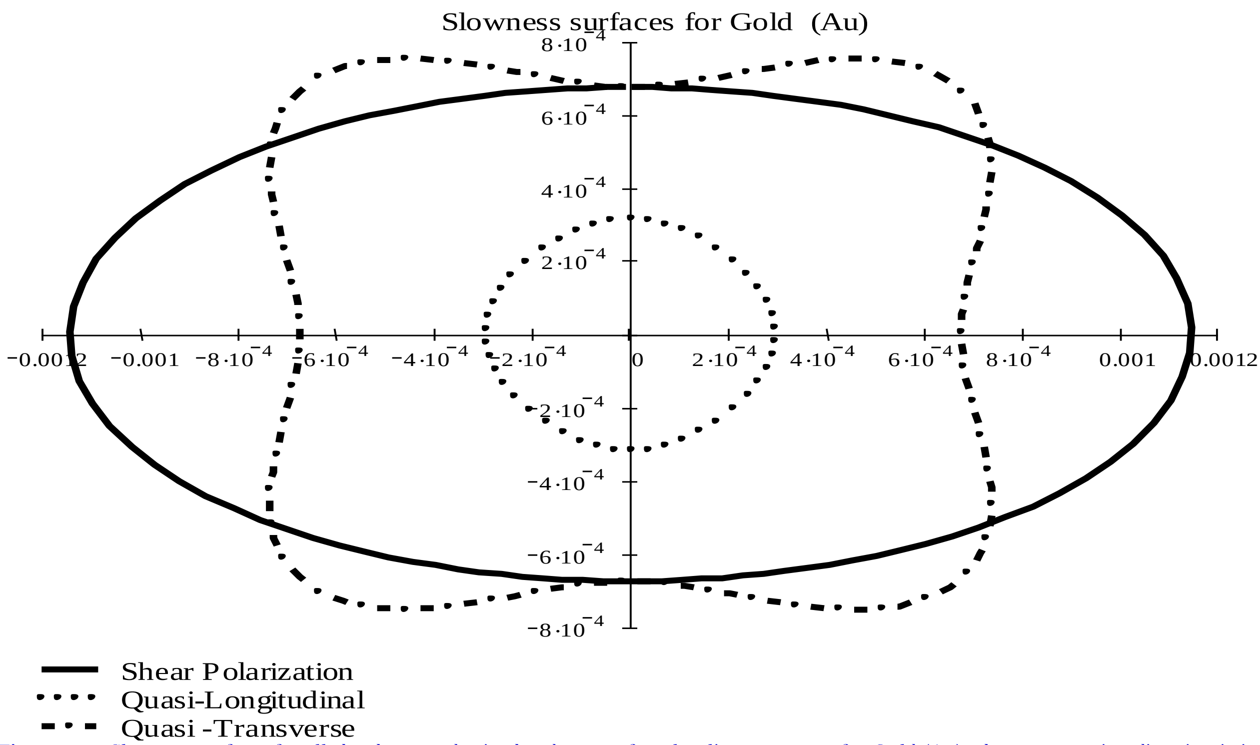

Figure-18. Slowness surfaces for all the three modes in the absence of nonlocality parameter for Gold (Au) when propagation direction is in Propagation in a diagonal plane.

7. CONCLUSIONS

Various features of the slowness or wave surface have been well acknowledged and qualitatively understood in classical (local) elastic solids, a very few study has been accomplished in nonlocal elasticity. Analytical scheme to determine the wave surface in general is based on the Christoffel equation [33] has been undertaken in this article we have derived the nonlocal Christoffel equation for anisotropic material, and then specializing it to the material of cubic symmetry. On the basis of this equation the following conclusions are drawn:

(i). A general nonlocal Christoffel equation is derived, eigenvalue solution is performed on the on this equation for materials of cubic symmetry.

(ii). Propagation along a cube face transverse wave polarized along OX_3, with velocity $c_t = \sqrt{C_{44}/\rho\left(1+\varepsilon^2\xi^2\right)}$, independent of the angle ϕ between the propagation direction $\mathbf{n}$ and the axis of the crystal, influenced by the non local parameter (ε) for any propagation direction in the (001) plane.

(iii). When $\phi=0$ or $\phi=\pi/2$ refers to a pure longitudinal, transverse waves in nonlocal elasticity. In all other directions, these waves' remains coupled and give velocities for quasi-longitudinal and quasi-transverse waves.

(iv). Propagation in a diagonal plane, although longitudinal and transverse waves travel on mutually perpendicular directions, one being quasi-transverse, and the other quasi-longitudinal waves, are influenced by the nonlocality parameter (ε).

(v). Phase and group wave velocities for longitudinal and transverse modes are influenced by the nonlocality parameter only when its value is greater than 0.001. Numerical calculation for crystals **Silicon(Si),** Aluminum (Al), Copper (Cu), Nickel (Ni), Gold (Au) are carried and found that velocities of longitudinal and transverse waves continuously decreases with increases of non-locality parameter.

(vi). Phase and group velocities are same when the nonlocal parameter is set equal to zero Figure 14, exhibiting that waves are non-dispersive in its counterpart local theory of elasticity.

(vii). When $\phi=0$ or $\phi=\dfrac{\pi}{2}$, from the slowness surfaces, waves are polarized and are dispersive one corresponds to pure longitudinal and two waves corresponds to a pure transverse also they are influenced by the nonlocality parameter (ε).

(viii). Anisotropy factor for crystals of a cubic symmetry is not influenced by the non local parameter.

It is expected to continue this study of this concepts and the development of computational tools that simulate wave propagation in general anisotropic media.

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