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SOME RELATIONS ON GEGENBAUER MATRIX POLYNOMIALS

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ABSTRACT

The main aim of this paper is devoted to derive some relations of Gegenbauer matrix polynomials of two variables. Volterra integral equation and a new representation of these matrix polynomials are given here. We introduce new generalized various forms of Gegenbauer matrix polynomials of two and three matrices by using the method of integral transforms to Hermite matrix polynomials. Furthermore, families of generating matrix functions are obtained and their applications are presented.

Keywords: Gegenbauer and hermite matrix polynomials, Matrix differential equations, Volterra integral equation, Integral representations, Generating matrix functions.

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Contribution/ Originality

This study is one of very few studies which have been investigated to new families of Gegenbauer matrix polynomials of two variables with two and three matrices is introduced using integral transform method and allow the derivation of a wealth of relations involving these matrix polynomials, and discuss its various special cases and their applications are presented. These results allow us to note that the use of the method of the integral representation is a fairly important tool of analysis and can be usefully extended to other families of Gegenbauer matrix polynomials which is a problem for further work.

1. INTRODUCTION AND DEFINITIONS

The study of special matrix polynomials is important due to their applications in various critical areas of statistics, physics, engineering and applied mathematics [1, 2]. Orthogonal matrix polynomials are becoming more and more relevant in the two preceding decades. Only very recently, some results in the theory of classical orthogonal polynomials are extended to orthogonal matrix polynomials; see [3-24].

In mathematics, Gegenbauer polynomials or ultraspherical polynomials $C_n^{\lambda}(x)$ are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$. They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi
polynomials. They are named after Leopold Gegenbauer.

In the scalar case, the Gegenbauer polynomials are defined by the generating function (see Andrews [25] pp.185 eq. (5.67) and Temme [26] pp.155 eq. (6.45))

$$F(x,t) = (1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n; \quad |x| \le 1, |t| < 1,$$

where $\lambda > -\frac{1}{2}$. By expanding the function $F(x,t) = (1-2xt+t^2)^{-\lambda}$ in a binomial series and we know that $F(t)$ has a convergent power series $\sum_{n=0}^{\infty} c_n t^n$ for $|t| < 1$.

Motivated by the work of Dattoli, et al. [27] who have used the link between Hermite and Gegenbauer polynomials to introduce generalized forms of Gegenbauer polynomials where the strategy of generalization outlined in Dattoli, et al. [27] benefits from the variety of existing Hermite matrix polynomials. The structure of this paper is as follows: In Section 2 summarizes previous results of Gegenbauer matrix polynomials of two variables and includes a new property of these matrix polynomials and gives a new generalization of the Gegenbauer matrix polynomials by means of the hypergeometric matrix function. Section 3, we build the Volterra integral equation of Gegenbauer matrix polynomials of two variables. In Section 4, we introduce a generalization for Gegenbauer matrix polynomials of two and three matrices by modifying the integral transform and give an explicit expression and generating matrix functions, which allow us to express them in terms of Hermite matrix polynomials. Some special cases of the results presented in this study are also indicated.

In this section, we will give some useful definitions, fact and lemmas. Throughout this paper, if A is a matrix in $C^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. The zero matrix or null matrix of $C^{N\times N}$ will be denoted by 0. Furthermore, the identity matrix of $C^{N\times N}$ will be denoted by I.

Fact 1.1. (Dunford and Schwartz [28])

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open region Ω of the complex plane, and A, B are matrices in $C^{N\times N}$ with $\sigma(A)\subset\Omega$ and $\sigma(B)\subset\Omega$, such that $AB = BA$, then from the properties of the matrix functional calculus, it follows *that*

$$f(A)g(B) = g(B)f(A). \qquad (1.1)$$

Definition 1.2. (Jódar and Cortés [29])

Let A *be a matrix in* $C^{N\times N}$, *the Pochhammer symbol or shifted factorial is defined*

$$(A)_n = A(A+I)(A+2I)...(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A);\, n \geq 1,\, (A)_0 = I. \qquad (1.2)$$

Lemma 1.3. (Jódar and Cortés [29])

If y *is a complex variable with* $|y| < 1$ *and* a *is a complex variable, then* $g(a) = (1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} y^n$ *is holomorphic function in* C. *Therefore, applying the holomorphic functional calculus [28] to any matrix* A *in* $C^{N\times N}$, *the image of* g, *acting on* A *yields*

$$g(A) = (1-y)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} y^n; \;\; |y| < 1, \qquad (1.3)$$

where $(A)_n$ is defined by (1.2).

Lemma 1.4. (Lancaster [30])

If $\|A\|$ *denotes any matrix norm for which* $\|I\| = 1$ *and if* A *is a matrix in* $C^{N\times N}$ *and* $\|A\| < 1$ *then* $(I+A)^{-1}$ *exists:*

$$(I+A)^{-1} = I - A + A^2 - A^3 + A^4 - A^5 + ... \qquad (1.4)$$

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z, then $z^{\frac{1}{2}}$ represents $\exp(\frac{1}{2}\log(z))$. If A is a matrix in $C^{N\times N}$ with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = \exp(\frac{1}{2}\log(A))$ denotes the image by $z^{\frac{1}{2}} = \sqrt{z} = \exp(\frac{1}{2}\log(z))$ of the matrix functional calculus is acting on the matrix A.

Definition 1.5. (Jódar and Company [31])

Let A *be a positive stable matrix in* $C^{N\times N}$ *such that*

$$Re(z) > 0, \quad \text{for every eigenvalue } z \in \sigma(A). \qquad (1.5)$$

Then the Hermite matrix polynomials are defined by

$$H_n(x, A) = n!\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}. \qquad (1.6)$$

Definition 1.6. (Jódar, et al. [32])

Let A *be a matrix in* $C^{N\times N}$ *satisfying the condition*

$$(-\frac{z}{2}) \notin \sigma(A) \text{ for all } z \in Z^+ \cup \{0\}. \qquad (1.7)$$

Then the Gegenbauer matrix polynomials are defined by

$$C_n^A(x) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}(A)_{n-k}. \qquad (1.8)$$

Definition 1.7. (Jódar and Cortés [2])

Let A *be a positive stable matrix in* $C^{N\times N}$*, then Gamma matrix function* $\Gamma(A)$ *is defined by*

$$\Gamma(A) = \int_0^\infty e^{-t} t^{A-I} dt; \quad t^{A-I} = \exp((A-I)\ln t). \qquad (1.9)$$

Notation 1.8. (Defez and Jódar [10])

By using (1.2), we have the relations

$$\begin{aligned} (A)_{n+k} &= (A)_n (A+nI)_k, \\ (A)_{2k} &= 2^{2k}(\frac{1}{2}A)_k(\frac{1}{2}(A+I))_k, \\ (-nI)_k &= \frac{(-1)^k\ n!}{(n-k)!} I, 0 \le k \le n. \end{aligned} \qquad (1.10)$$

We conclude this section recalling a result related to the rearrangement of the terms in an iterated series. If $A(k,n)$ and $B(k,n)$ are matrices in $C^{N\times N}$ for $n \ge 0$, $k \ge 0$, then in an analogous way to the proof of Lemma 11 (see,Defez and Jódar [1]), it follows that

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} A(k,n-2k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} A(k,n-mk); m \in N. \end{aligned} \qquad (1.11)$$

Similarly to (1.11), we can write

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n+k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk); m \in N \end{aligned} \qquad (1.12)$$

where $[x]$ denotes the greatest integer in x .

Recently, in Shehata [33], a new extension of Gegenbauer matrix polynomials with two variables was presented. To defined it, the starting point was the following formula

$$F(x,y,t,A)=(1-2xt+yt^2)^{-A}=\sum_{n=0}^{\infty}C_n^A(x,y)t^n;\quad |x|\leq 1,|y|\leq 1,|t|<1. \qquad (1.13)$$

This formula turns out to be the key for the definition and development of the properties mentioned in the paper [33], to guarantee that (1.13) is term-wise differentiable with respect to its variables x , y and t . However, we will see that formula (1.13) is correct with the addition of the following conditions $|2xt-yt^2|<1$ and $\left|\frac{yt}{2x}\right|<1$. This is to clarify the correct definition of the generating matrix function for Gegenbauer matrix polynomials.

2.PROPERTIES OF GEGENBAUER MATRIX POLYNOMIALS

Let A be a matrix in $C^{N\times N}$ satisfying the spectral condition $(-\frac{1}{2}z)\notin\sigma(A)$ $\forall z\in Z^+\cup\{0\}$. If r_1 and r_2 are the roots of the quadratic equation $1-2xt+yt^2=0$ and r is the minimum of the set $\{r_1,r_2\}$, then the matrix function $F(x,y,t,A)$ regarded as a function of t , is analytic in the disk $|t|<r$ for every real number in $|x|\leq 1$ and $|y|\leq 1$. Therefore, Gegenbauer matrix polynomials of two variables are defined by the generating matrix function [33]

$$F(x,y,t,A)=(1-2xt+yt^2)^{-A}=\sum_{n=0}^{\infty}C_n^A(x,y)t^n;\quad |x|\leq 1,|y|\leq 1,|t|<r \qquad (2.1)$$

or, equivalently to the previous definition to generating matrix function in (1.13). Then, the Gegenbauer matrix polynomials of two variables are defined by

$$C_n^A(x,y)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k y^k(2x)^{n-2k}}{k!(n-2k)!}(A)_{n-k}. \qquad (2.2)$$

It is clear that

$$C_0^A(x,y)=I,\ C_1^A(x,y)=2xA,$$

$$C_2^A(x,y)=2x^2A(A+I)-yA \text{ and } C_n^A(x,0)=\frac{(2x)^n}{n!}(A)_n.$$

Replacing x by $\frac{x}{\sqrt{y}}$ and t by $t\sqrt{y}$ in (2.1) and in view of (1.8), we get

$$(1-2xt+yt^2)^{-A} = \sum_{n=0}^{\infty} C_n^A\left(\frac{x}{\sqrt{y}}\right)(t\sqrt{y})^n. \qquad (2.3)$$

Now, comparing (2.1) with (2.3), we obtain

$$\sum_{n=0}^{\infty} C_n^A(x,y)t^n = \sum_{n=0}^{\infty} C_n^A\left(\frac{x}{\sqrt{y}}\right)(t\sqrt{y})^n.$$

Equating the Coefficients of t^n, we get

$$C_n^A(x,y) = y^{\frac{n}{2}} C_n^A\left(\frac{x}{\sqrt{y}}\right). \qquad (2.4)$$

Replacing x by $-x$ and t by $-t$ in (2.1), the left side remains unchanged, we obtain

$$C_n^A(-x,y) = (-1)^n C_n^A(x,y). \qquad (2.5)$$

For $x=1$ and $y=1$, we have

$$(1-t)^{-2A} = \sum_{n=0}^{\infty} t^n C_n^A(1,1); |t|<1.$$

By (1.3) to obtain

$$C_n^A(1,1) = \frac{1}{n!}(2A)_n. \qquad (2.6)$$

For $x=0$, it follows

$$(1+yt^2)^{-A} = \sum_{n=0}^{\infty} t^n C_n^A(0,y).$$

Also, by (1.3) one gets

$$(1+yt^2)^{-A} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} y^n t^{2n} (A)_n; |yt^2|<1.$$

Therefore, we have

$$C_{2n}^A(0,y) = \frac{(-1)^n}{n!} y^n (A)_n, C_{2n+1}^A(0,y) = 0. \qquad (2.7)$$

From (2.7) one gets

$$\frac{\partial}{\partial x} C_{2n}^A(0,y) = 0, \frac{\partial}{\partial x} C_{2n+1}^A(0,y) = \frac{2(-1)^n}{n!} y^n A(A+I)_n. \qquad (2.8)$$

For $y=1$, (2.4) reduces to

$$C_n^A(x,1) = C_n^A(x) \qquad (2.9)$$

where $C_n^A(x)$ is well known Gegenbauer matrix polynomials [15, 34].

Note that the Gegenbauer's matrix polynomials of two variables are the solutions of the following matrix partial differential equation:

$$(y-x^2)\frac{\partial^2}{\partial x^2}C_n^A(x,y)-x(2A+I)\frac{\partial}{\partial x}C_n^A(x,y)+n(2A+nI)C_n^A(x,y)=0, n\geq 0. \quad (2.10)$$

and satisfy the three terms matrix recurrence relationship:

$$nC_n^A(x,y)=2x(A+(n-1)I)C_{n-1}^A(x,y)-y(2A+(n-2)I)C_{n-2}^A(x,y); n\geq 2. \quad (2.11)$$

We recall that for the Gegenbauer matrix polynomials of two variables the following relations are obtained

$$\frac{\partial^r}{\partial x^r}C_n^A(x,y)+(-1)^{r-1}2^r\frac{\partial^r}{\partial y^r}C_{n+r}^A(x,y)=0 \quad (2.12)$$

and

$$\begin{aligned}\frac{\partial^r}{\partial x^r}C_n^A(x,y)&=2^r(A)_r\, C_{n-r}^{A+rI}(x,y),\\ \frac{\partial^r}{\partial y^r}C_n^A(x,y)&=(-1)^r(A)_r\, C_{n-2r}^{A+rI}(x,y).\end{aligned} \quad (2.13)$$

From (2.1) and using the relations (1.10) with $|t\sqrt{y}|<1$ and $\left|\frac{2t(x-\sqrt{y})}{(1-t\sqrt{y})^2}\right|<1$, we have

$$\begin{aligned}(1-2xt+(t\sqrt{y})^2)^{-A}&=(1-\frac{2t(x-\sqrt{y})}{(1-t\sqrt{y})^2})^{-A}(1-t\sqrt{y})^{-2A}\\ &=\sum_{k=0}^{\infty}\frac{(A)_k 2^k t^k(x-\sqrt{y})^k}{k!}(1-t\sqrt{y})^{-(2A+2kI)}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(A)_k(2A+2kI)_n 2^k t^k(x-\sqrt{y})^k}{n!k!}(t\sqrt{y})^n\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(A)_k(2A)_{n+2k}[(2A)_{2k}]^{-1}2^k(x-\sqrt{y})^k}{n!k!}(\sqrt{y})^n t^{n+k}\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(A)_k(2A)_n(2A+nI)_k[(2A)_{2k}]^{-1}2^k(x-\sqrt{y})^k}{(n-k)!k!}(\sqrt{y})^{n-k}t^n\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(-nI)_k(2A)_n(2A+nI)_k[(A+\frac{1}{2}I)_k]^{-1}\left(\frac{x-\sqrt{y}}{2\sqrt{y}}\right)^k}{n!k!}(\sqrt{y})^n t^n\end{aligned}$$

These results are summarized below.

Theorem 2.1.

Let A *be a matrix in* $C^{N\times N}$ *such that* A *satisfy the condition* $(-\frac{1}{2}z)\notin\sigma(A)$,

$\forall z \in Z^{+} \cup \{0\}$ *with* $\left| \dfrac{\sqrt{y}-x}{2\sqrt{y}} \right| < 1$. *Then the Gegenbauer matrix polynomials of two variables have the following hypergeometric matrix representation:*

$$C_n^A(x,y) = \frac{(2A)_n y^{\frac{n}{2}}}{n!} \, {}_2F_1\left(-nI, 2A+nI; A+\frac{1}{2}I; \frac{\sqrt{y}-x}{2\sqrt{y}} \right). \qquad (2.14)$$

Definition 2.2.(Defez and Jódar [1])

Let B *be a matrix in* $C^{N\times N}$ *such that*

$$B+nI \quad \text{is an invertible matrix for all integers } n \geq 0. \qquad (2.15)$$

Then, we define a new generalized Gegenbauer matrix polynomials of two matrices by using the hypergeometric matrix function in the form:

$$C_n^A(x,y;B) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k}{k!(n-2k)!} (2x)^{n-2k} \mathrm{B}_{n,k} A_{n-k} \qquad (2.16)$$

where

$$\mathrm{B}_{n,k} = n! \, y^k \; {}_2F_1(-kI, B; -nI; y) = \sum_{i=0}^{k} \frac{k!(n-i)!}{i!(k-i)!} (B)_i \, y^{k+i}, \qquad (2.17)$$

where A and B are matrices in $C^{N\times N}$ such that A satisfies the condition (1.7), B satisfies the condition (2.15) and $AB = BA$.

When B is the zero matrix, then the generalized Gegenbauer matrix polynomials of two variables reduce to

$$C_n^A(x,y;0) = C_n^A(x,y) \qquad (2.18)$$

From (2.17), we can write in the following integral representation

$$\mathrm{B}_{n,k} = y^k \Gamma^{-1}(B) \int_0^\infty \int_0^\infty e^{-(t+u)} t^n u^{B-I} \left(1+\frac{yu}{t} \right)^k dtdu. \qquad (2.19)$$

Theorem 2.3.

Let A *and* B *be matrices in* $C^{N\times N}$ *such that* A *satisfy the condition (1.7) and* B *satisfy the condition (2.15). Then the generalized Gegenbauer matrix polynomials of two variables have the following integral representation:*

$$C_n^A(x,y;B) = \Gamma^{-1}(B) \int_0^\infty \int_0^\infty e^{-(t+u)} t^n u^{B-I} C_n^A\left(x, y(1+\frac{yu}{t}) \right) dtdu. \qquad (2.20)$$

Proof. Using (2.16), (2.17) and (2.19), we obtain (2.20). Thus the proof is completed.

The purpose of the next section is to introduce the Gegenbauer matrix polynomials of two variables by means of a Volterra integral equation.

3.VOLTERRA INTEGRAL EQUATION

Replacing n by $2n$ in Eq. (2.10), we write the matrix differential equation

$$(y-x^2)\frac{\partial^2}{\partial x^2}C_{2n}^{A}(x,y)-x(2A+I)\frac{\partial}{\partial x}C_{2n}^{A}(x,y)+2n(2A+2nI)C_{2n}^{A}(x,y)=0. \quad (3.1)$$

From the equation (2.17), one gets

$$\frac{\partial^2}{\partial x^2}C_{2n}^{A}(x,y)=2A\frac{\partial}{\partial x}C_{2n-1}^{A+I}(x,y)=4A(A+I)\,C_{2n-2}^{A+2I}(x,y). \quad (3.2)$$

Integrating (3.2) and using (2.7) and (2.8), we get

$$\frac{\partial}{\partial x}C_{2n}^{A}(x,y)=4A(A+I)\int_0^x C_{2n-2}^{A+2I}(z,y)dz \quad (3.3)$$

and

$$C_{2n}^{A}(x,y)=4A(A+I)\int_0^x (x-z)C_{2n-2}^{A+2I}(z,y)dz+\frac{(-1)^n}{n!}y^n(A)_n. \quad (3.4)$$

Using (3.2)-(3.4) in (3.1) gives

$$\begin{aligned}&(y-x^2)C_{2n-2}^{A+2I}(x,y)-x(2A+I)\int_0^x C_{2n-2}^{A+2I}(z,y)dz\\&+2n(2A+2nI)\int_0^x (x-z)C_{2n-2}^{A+2I}(z,y)dz+\frac{(-1)^n}{(n-1)!}y^n(A+nI)(A+2I)_n=0.\end{aligned} \quad (3.5)$$

From (3.5) and we replace n by $n+1$, we get Replacing n by $n+1$ in Eq. (3.5), we have

$$\begin{aligned}&(y-x^2)C_{2n}^{A+2I}(x,y)-x(2A+I)\int_0^x C_{2n}^{A+2I}(z,y)dz\\&+4(n+1)(A+(n+1)I)\int_0^x (x-z)C_{2n}^{A+2I}(z,y)dz\\&+\frac{(-1)^{n+1}}{n!}y^{n+1}(A+(n+1)I)(A+2I)_{n+1}=0.\end{aligned} \quad (3.6)$$

Next, the matrix differential equation (2.4) with $n=2n+1$ can be expressed as

$$\begin{aligned}&(y-x^2)\frac{\partial^2}{\partial x^2}C_{2n+1}^{A}(x,y)-x(2A+I)\frac{\partial}{\partial x}C_{2n+1}^{A}(x,y)\\&+(2n+1)(2A+(2n+1)I)C_{2n+1}^{A}(x,y)=0.\end{aligned} \quad (3.7)$$

The formula (2.6) gives

$$\frac{\partial^2}{\partial x^2}C_{2n+1}^{A}(x,y)=2A\frac{\partial}{\partial x}C_{2n}^{A+I}(x,y)=4A(A+I)\,C_{2n-1}^{A+2I}(x,y). \quad (3.8)$$

By integrating (3.8) with the help of (2.12) and (2.13) we get

$$\frac{\partial}{\partial x}C_{2n+1}^{A}(x,y)=4A(A+I)\int_0^x C_{2n-1}^{A+2I}(z,y)dz. \quad (3.9)$$

and

$$C_{2n+1}^{A}(x,y) = 4A(A+I)\int_0^x (x-z)C_{2n-1}^{A+2I}(z,y)dz + \frac{2(-1)^n}{n!}xy^n A(A)_n. \qquad (3.10)$$

By virtue of Eqs. (3.8)-(3.10), the matrix differential equation (3.7) becomes

$$\begin{aligned}&(y-x^2)C_{2n-1}^{A+2I}(x,y) - x(2A+I)\int_0^x C_{2n-1}^{A+2I}(z,y)dz\\ &+(2n+1)(2A+(2n+1)I)\int_0^x (x-z)C_{2n-1}^{A+2I}(z,y)dz\\ &+\frac{2(-1)^n(n+\frac{1}{2})}{n!}xy^n A(A+(n+\frac{1}{2})I)(A+2I)_n = 0\end{aligned} \qquad (3.11)$$

which on replacing n by $n+1$ in Eq. (3.11) implies

$$\begin{aligned}&(y-x^2)C_{2n+1}^{A+2I}(x,y) - x(2A+I)\int_0^x C_{2n+1}^{A+2I}(z,y)dz\\ &+(2n+3)(2A+(2n+3)I)\int_0^x (x-z)C_{2n+1}^{A+2I}(z,y)dz\\ &+\frac{2(-1)^{n+1}(n+\frac{3}{2})}{(n+1)!}xy^{n+1} A(A+(n+\frac{3}{2})I)(A+2I)_{n+1} = 0.\end{aligned} \qquad (3.12)$$

Combining Eqs. (3.6) and (3.12) demonstrates in the following result:

Theorem-3.1.

Let A be a matrix in satisfying the condition (1.7). The Volterra integral equation of the Gegenbauer matrix polynomials of two variables is given as

$$\begin{aligned}&(y-x^2)C_{n}^{A+2I}(x,y) - x(2A+I)\int_0^x C_{n}^{A+2I}(z,y)dz\\ &+(n+2)(2A+(n+2)I)\int_0^x (x-z)C_{n}^{A+2I}(z,y)dz\\ &+\frac{(-1)^{m+1}(\frac{1}{2}n+1)}{(m+1)!}(2xA)^{n-2m}y^{m+1}(A+(\frac{1}{2}n+1)I)(A+2I)_{m+1} = 0,\end{aligned} \qquad (3.13)$$

where $n = 2m$.

In the next section, we introduce new families of Gegenbauer matrix polynomials of two and three matrices by using the integral representation method. The generating matrix functions, matrix partial differential equation and the series definition for the new families of Gegenbauer matrix polynomials are derived. We also give here the finite summation formula involving their Gegenbauer matrix polynomials and discuss its various special cases.

4.CONNECTIONS BETWEEN GEGENBAUER AND HERMITE MATRIX POLYNOMIALS

As already remarked, firstly, a new generalized form of the Gegenbauer matrix polynomials of two variables is introduced by using the integral representation method

$$C_n^A(x,y;B)=\frac{1}{n!}\Gamma^{-1}(A)\int_0^\infty e^{-t}t^{A+(n-1)I}\ H_n\left(x,\frac{y}{t},B\right)dt \tag{4.1}$$

where Hermite matrix polynomials of two variables $H_n(x,y,B)$ defined by Batahan [35]

$$H_n(x,y,B)=n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k y^k}{k!(n-2k)!}(x\sqrt{2B})^{n-2k},n\geq 0. \tag{4.2}$$

It is evident that in view of the relations

$$\begin{aligned}H_n\left(x,\frac{y}{t},B\right)&=t^{-\frac{n}{2}}H_n(x\sqrt{t},y,B),\\ H_n\left(x,\frac{y}{t},B\right)&=t^{-n}H_n(xt,yt,B).\end{aligned} \tag{4.3}$$

From (4.1) and (4.3), we can be expressed equivalently as

$$C_n^A(x,y;B)=\frac{1}{n!}\Gamma^{-1}(A)\int_0^\infty e^{-t}t^{A+(\frac{n}{2}-1)I}\ H_n(x\sqrt{t},y,B)dt. \tag{4.4}$$

Also, in view of the relation (4.3) the above equation can be expressed equivalently as

$$C_n^A(x,y;B)=\frac{1}{n!}\Gamma^{-1}(A)\int_0^\infty e^{-t}t^{A-I}\ H_n(xt,yt,B)dt. \tag{4.5}$$

Now, making use of equation (1.9), (4.2) and (4.5) we find that the Gegenbauer matrix polynomials of two variables are defined by the following series

$$C_n^A(x,y;B)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k y^k (A)_{n-k}}{k!(n-2k)!}(x\sqrt{2B})^{n-2k}. \tag{4.6}$$

We have the following main theorem.

Theorem 4.1.

Let B *be a matrix in* $C^{N\times N}$, *where* $Re(\mu)>0$ *for all eigenvalues* $\mu\in\sigma(B)$, *and let* A *be a matrix in* $C^{N\times N}$ *satisfying the condition* $(-\frac{z}{2})\notin\sigma(A)$ *for all* $z\in Z^+\cup\{0\}$, $AB=BA$ *and* $\|B\|<\frac{1}{\sqrt{2}}$. *Then the generating matrix function for Gegenbauer matrix polynomials of two matrices is*

$$\sum_{n=0}^{\infty}C_n^A(x,y;B)t^n=(I-xt\sqrt{2B}+yt^2I)^{-A} \tag{4.7}$$

where $\left\| xt\sqrt{2B} - yt^2 I \right\| < 1$.

Proof. From (1.12) and (4.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} C_n^A(x,y;B)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k y^k (x\sqrt{2B})^{n-2k}(A)_{n-k}}{k!(n-2k)!} t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(A)_n (-y)^k (x\sqrt{2B})^{n-k}}{k!(n-k)!} t^{n+k}.\end{aligned}$$

Using (1.2) and (1.3), one get

$$(x\sqrt{2B} - ytI)^n = \sum_{k=0}^{n} \frac{(-yt)^k (x\sqrt{2B})^{n-k}}{k!(n-k)!}$$

we can write

$$\sum_{n=0}^{\infty} C_n^A(x,y;B)t^n = \sum_{n=0}^{\infty} \frac{(A)_n (x\sqrt{2B} - ytI)^n t^n}{n!} = (I - xt\sqrt{2B} + yt^2 I)^{-A}$$

This completes the proof.

Secondly, we can introduce generalized Gegenbauer matrix polynomials, the new Gegenbauer-type matrix polynomials of three matrices defined by using the integral representation in the following relations:

$$C_n^A(x,y;B,P) = \frac{1}{n!}\Gamma^{-1}(A)\int_0^{\infty} e^{-Pt} t^{A+(\frac{n}{2}-1)I}\, H_n(x\sqrt{t}, y, B)dt. \tag{4.8}$$

Also, in view of the relation (4.8) the above equation can be expressed equivalently as

$$C_n^A(x,y;B,P) = \frac{1}{n!}\Gamma^{-1}(A)\int_0^{\infty} e^{-Pt} t^{A+(n-1)I}\, H_n\left(x, \frac{y}{t}, B\right)dt \tag{4.9}$$

and

$$C_n^A(x,y;B,P) = \frac{1}{n!}\Gamma^{-1}(A)\int_0^{\infty} e^{-Pt} t^{A-I}\, H_n(xt, yt, B)dt. \tag{4.10}$$

By using Hermite matrix polynomials of two variables and (4.10), we obtain the Gegenbauer matrix polynomials of two variables $C_n^A(x,y;B,P)$

$$C_n^A(x,y;B,P) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k P^{(k-n)I-A}(A)_{n-k}}{k!(n-2k)!} y^k (x\sqrt{2B})^{n-2k}. \tag{4.11}$$

Theorem 4.2.

Let B *and* P *be positive stable matrices in* $C^{N\times N}$ *satisfying the conditions* $Re(\mu)>0$ *for all eigenvalues* $\mu\in\sigma(B)$, $Re(\nu)>0$ *for all eigenvalues* $\nu\in\sigma(P)$, *and let* A *be a matrix in* $C^{N\times N}$ *satisfying the condition* $(-\frac{z}{2})\notin\sigma(A)$ *for all* $z\in Z^{+}\cup\{0\}$, *such that* A ,B *and* P *are commuting matrices and* $\|B\|<\frac{1}{\sqrt{2}}$ *. Then the Gegenbauer matrix polynomials of two variables have the generating matrix function:*

$$\sum_{n=0}^{\infty}C_n^A(x,y;B,P)t^n=(P-xt\sqrt{2B}+yt^2I)^{-A} \qquad (4.12)$$

where $\left\|xt\sqrt{2B}-yt^2I\right\|<\|P\|$.

Proof. This can be proved by using equations (1.3), (1.4), (1.12) and (4.10).

Similarly, thirdly, we introduce the extension Gegenbauer matrix polynomials of two variables using the Hermite matrix polynomials to get further extensions in the following form

$$C_{n,m}^A(x,y;B,P)=\frac{1}{n!}\Gamma^{-1}(A)\int_0^{\infty}e^{-Pu}u^{A+(n-1)I}\ H_{n,m}\left(x,\frac{y}{u^{m-1}},B\right)du, \qquad (4.13)$$

we can immediately derive that

$$C_{n,m}^A(x,y;B,P)=\frac{1}{n!}\Gamma^{-1}(A)\int_0^{\infty}e^{-Pu}u^{A-I}\ H_{n,m}(xu,yu,B)du. \qquad (4.14)$$

where 2-index Hermite matrix polynomials of two variables $H_{n,m}(x,y,B)$ are specified by the series definition [36].

$$H_{n,m}(x,y,B)=n!\sum_{k=0}^{[\frac{1}{m}n]}\frac{(-1)^k y^k}{k!(n-mk)!}(x\sqrt{mB})^{n-mk}. \qquad (4.15)$$

By using (4.14) and (4.15), we obtain the series definition for the generalized Gegenbauer matrix polynomials of two variables as

$$C_{n,m}^A(x,y;B,P)=\sum_{k=0}^{[\frac{1}{m}n]}\frac{(-1)^k y^k P^{(m-1)kI-nI-A}(A)_{n-(m-1)k}}{k!(n-mk)!}(x\sqrt{mB})^{n-mk}. \qquad (4.16)$$

Theorem 4.3.

Let B *and* P *be positive stable matrices in* $C^{N\times N}$ *satisfying the conditions* $Re(\mu)>0$ *for all eigenvalues* $\mu\in\sigma(B)$, $Re(\nu)>0$ *for all eigenvalues* $\nu\in\sigma(P)$, *and let* A *be a matrix in* $C^{N\times N}$ *satisfying the condition* $(-\frac{z}{2})\notin\sigma(A)$ *for all* $z\in Z^{+}\cup\{0\}$, *such that* A ,B *and* P *are* *commuting matrices and* $\|B\| < \dfrac{1}{\sqrt{m}}$. *Then we have:*

$$\sum_{n=0}^{\infty} C_{n,m}^{A}(x,y;B,P)t^{n} = (P - xt\sqrt{mB} + yt^{m}I)^{-A} \qquad (4.17)$$

where $\left\|xt\sqrt{mB} - yt^{m}I\right\| < \|P\|$.

Proof. From (1.3), (1.4), (1.12) and (4.16), we get the proof of equation (4.17) similarly.

In the following, we obtain some properties for the Gegenbauer matrix polynomials of two variables as follows.

Theorem 4.4.

The Gegenbauer matrix polynomials $C_{n,m}^{A}(x,y;B,P)$ *of two variables satisfying the following relation*

$$\frac{\partial^{r}}{\partial x^{r}} C_{n,m}^{A}(x,y;B,P) + (-1)^{r-1}(\sqrt{mB})^{r} \frac{\partial^{r}}{\partial y^{r}} C_{n+(m-1)r}^{A}(x,y;B,P) = 0. \qquad (4.18)$$

Proof. Differentiating (4.17) with respect to x and y, we get

$$tA\sqrt{mB}(P - xt\sqrt{mB} + yt^{m}I)^{-(A+I)} = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} C_{n,m}^{A}(x,y;B,P)t^{n} \qquad (4.19)$$

and

$$-t^{m}A(P - xt\sqrt{mB} + yt^{m}I)^{-(A+I)} = \sum_{n=0}^{\infty} \frac{\partial}{\partial y} C_{n,m}^{A}(x,y;B,P)t^{n}. \qquad (4.20)$$

Iteration (4.19) and (4.20), for $0 \le r \le n$, implies (4.18) and the proof is completed.

Theorem-4.5.

The Gegenbauer matrix polynomials $C_{n,m}^{A}(x,y;B,P)$ *of two variables satisfying the following relations*

$$\begin{aligned} \frac{\partial^{r}}{\partial x^{r}} C_{n,m}^{A}(x,y;B,P) &= (\sqrt{mB})^{r}(A)_{r}\, C_{n-(m-1)r,m}^{A+rI}(x,y;B,P), \\ \frac{\partial^{r}}{\partial y^{r}} C_{n,m}^{A}(x,y;B,P) &= (-1)^{r}(A)_{r}\, C_{n-mr,m}^{A+rI}(x,y;B,P). \end{aligned} \qquad (4.21)$$

Proof. From (4.19) and (4.20), we can write

$$\begin{aligned}
\sqrt{mB}A(P-2xt\sqrt{mB}+yt^{m}I)^{-(A+I)} &= \sum_{n=1}^{\infty}\frac{\partial}{\partial x}C_{n,m}^{A}(x,y;B,P)t^{n-1} \\
&= \sum_{n=0}^{\infty}\frac{\partial}{\partial x}C_{n+1,m}^{A}(x,y;B,P)t^{n}, \\
-A(I-xt\sqrt{mB}+yt^{m}I)^{-(A+I)} &= \sum_{n=m}^{\infty}\frac{\partial}{\partial y}C_{n,m}^{A}(x,y;B,P)t^{n-m} \\
&= \sum_{n=0}^{\infty}\frac{\partial}{\partial y}C_{n+m,m}^{A}(x,y;B,P)t^{n}.
\end{aligned} \tag{4.22}$$

By applying (4.17), it follows that

$$\begin{aligned}
A\sqrt{mB}(P-xt\sqrt{mB}+yt^{m}I)^{-(A+I)} &= \sum_{n=0}^{\infty}A\sqrt{mB}\,C_{n,m}^{A+I}(x,y;B,P)t^{n}, \\
-A(P-xt\sqrt{mB}+yt^{m}I)^{-(A+I)} &= -\sum_{n=0}^{\infty}A\,C_{n,m}^{A+I}(x,y;B,P)t^{n}.
\end{aligned} \tag{4.23}$$

Identification of the coefficients of t^{n} in (4.23) yields

$$\begin{aligned}
\frac{\partial}{\partial x}C_{n+1,m}^{A}(x,y;B,P) &= A\sqrt{mB}\,C_{n,m}^{A+I}(x,y;B,P), \\
\frac{\partial}{\partial y}C_{n+m,m}^{A}(x,y;B,P) &= -A\,C_{n,m}^{A+I}(x,y;B,P).
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{\partial}{\partial x}C_{n,m}^{A}(x,y;B,P) &= A\sqrt{mB}\,C_{n-1,m}^{A+I}(x,y;B,P), \\
\frac{\partial}{\partial y}C_{n,m}^{A}(x,y;B,P) &= -A\,C_{n-m,m}^{A+I}(x,y;B,P).
\end{aligned} \tag{4.24}$$

Iteration (4.24), for $0\le r\le n$, implies (4.21), we complete the proof.

Remark-4.1.

The generating matrix function for the Hermite matrix polynomials is

$$\lim_{r\to-\infty}\left(I-\frac{xt\sqrt{mB}}{r}+\frac{yt^{m}}{r}I\right)^{-r} = \exp[xt\sqrt{mB}-yt^{m}I] = \sum_{n=0}^{\infty}\frac{t^{n}}{n!}H_{n,m}(x,y,B).$$

As a result we obtain the following finite summation formula:

Theorem-4.6.

Gegenbauer and Hermite matrix polynomials satisfy

$$C_{n,m}^{A}(x,y;B,P) = \sum_{k=0}^{[\frac{1}{m}n]} \frac{(-1)^k (A)_{n-(m-1)k}}{k!(n-mk)!} \; {}_mF_0(-kI, \frac{A+(n-(m-1)k)I}{m-1},...,$$
$$\frac{A+(n-(m-1)k+(m-1)I)I}{m-1};-;((m-1)P^{-1})^{m-1}) \qquad (4.25)$$
$$2^n(\sqrt{2A})^{-n} y^k H_{n-mk,m}(x,y,B).$$

Proof. In Metwally, et al. [36], the expansion of $x^n I$ in a series of Hermite matrix polynomials has been given in the form

$$(x\sqrt{mB})^n = \sum_{r=0}^{[\frac{1}{m}n]} \frac{n!}{r!(n-mr)!} H_{n-mr}(x,y,B). \qquad (4.26)$$

By using (1.12), (4.16) and (4.26), it can be proved.

On the other hand, differentiating (4.16) with respect to x, the $C_{n,m}^{A}(x,y;B,P)$ satisfy differential recurrence relation

$$\frac{\partial}{\partial x} C_{n,m}^{A}(x,y;B,P) = A\sqrt{mB} C_{n-1,m}^{A+I}(x,y;B,P). \qquad (4.27)$$

The relation is of special interest, we can write

$$\frac{\partial}{\partial x} C_{n+1,m}^{A-I}(x,y;B,P) = (A-I)\sqrt{mB}\, C_{n,m}^{A}(x,y;B,P). \qquad (4.28)$$

Compare (4.19) with (4.18), we obtain

$$\frac{\partial^2}{\partial x^2} C_{n+1,m}^{A-I}(x,y;B,P) = A(A-I)(\sqrt{mB})^2\, C_{n-1,m}^{A+I}(x,y;B,P). \qquad (4.29)$$

In (4.20), it is easy to see

$$\frac{\partial^n}{\partial x^n} C_{n+k,m}^{A}(x,y;B,P) = (-\sqrt{mB})^n \Gamma(I-A)\Gamma^{-1}((1-n)I-A) C_{k,m}^{A-nI}(x,y;B,P). \qquad (4.30)$$

On the other hand, differentiating (4.17) partially with respect to x and then with respect to t, we get the following differential recurrence relations

$$(x\frac{\partial}{\partial x} - n) C_{n,m}^{A}(x,y;B,P) = y\frac{\partial}{\partial x} C_{n-m+1,m}^{A}(x,y;B,P) \qquad (4.31)$$

and

$$[(m-1)x\frac{\partial}{\partial x}I+nI+mA]C_{n,m}^{A}(x,y;B,P)=P\frac{\partial}{\partial x}C_{n+1,m}^{A}(x,y;B,P). \qquad (4.32)$$

Now iteratively applying the linear differential operator in (4.32) $m-1$ times on (4.31) and using at each step the simple relation

$$(ax\frac{\partial}{\partial x}+b)\frac{\partial}{\partial x}^{n}=\frac{\partial}{\partial x}^{n}(ax\frac{\partial}{\partial x}+b-na). \qquad (4.33)$$

From (4.33), we get the matrix differential equation of order m in the following form

$$\begin{aligned}&[((m-1)x\frac{\partial}{\partial x}I+nI+mA+m(m-2)I)((m-1)x\frac{\partial}{\partial x}I+nI+mA+m(m-3)I)\\&...((m-1)x\frac{\partial}{\partial x}I+nI+mA)(x\frac{\partial}{\partial x}-n)]C_{n,m}^{A}(x,y;B,P)\\&=ym^{m}(\sqrt{mB})^{-m}P^{m-1}\frac{\partial^{m}}{\partial x^{m}}C_{n,m}^{A}(x,y;B,P)\end{aligned} \qquad (4.34)$$

These results are summarized below.

Theorem-4.7.

Let A, B, and P be commuting matrices in $C^{N\times N}$. Then the $C_{n,m}^{A}(x,y;B,P)$ is a solution of the matrix differential equation of order m.

Now, by making use of the differential operator $\theta=x\frac{\partial}{\partial x}$ which possesses the interesting property that $\theta x^{n}=nx^{n}$, we have

$$\begin{aligned}(-\theta)_{m}C_{n,m}^{A}(x,y;B,P)&=\sum_{k=0}^{[\frac{1}{m}n]}\frac{(-1)^{k}y^{k}(-n+mk)_{m}P^{(m-1)kI-nI-A}(A)_{n-(m-1)k}}{k!(n-mk)!}(x\sqrt{mB})^{n-mk}\\&=\sum_{k=0}^{[\frac{1}{m}n]}\frac{(-1)^{k}y^{k}(-n)_{mk+m}P^{(m-1)kI-nI-A}(A)_{n-(m-1)k}}{k!(n-mk)!(-n)_{mk}}(x\sqrt{mB})^{n-mk}\end{aligned} \qquad (4.35)$$

and also

$$\begin{aligned}&(-\theta+n)(-\frac{m-1}{m}\theta I-\frac{n}{m}I-A-(m-2)I)_{m-1}C_{n,m}^{A}(x,y;B,P)\\&=\sum_{k=1}^{[\frac{1}{m}n]}\frac{(-1)^{k}y^{k}P^{(m-1)(k-1)I-nI-A}(A)_{n-(m-1)(k-1)}}{(k-1)!(n-mk)!}\sqrt{mB}(x\sqrt{mB})^{n-mk}\\&=myP^{m-1}(x\sqrt{mB})^{-m}(-\theta)_{m}C_{n,m}^{A}(x,y;B,P).\end{aligned} \qquad (4.36)$$

On the other hand, for $C^{A}_{n,m}(x,y;B,P)$, we derive the following relations:

$$
\begin{aligned}
&[(-\theta+n)(\frac{m-1}{m}\theta I-\frac{n}{m}I-A-(m-2)I)_{m-1}\\
&-myP^{m-1}(x\sqrt{mB})^{-m}(-\theta)_{m}]C^{A}_{n,m}(x,y;B,P)=0
\end{aligned}
\tag{4.37}
$$

where, by virtue of the formula $(-\theta)_{n}x^{k}=(-x)^{n}\dfrac{\partial^{n}}{\partial x^{n}}x^{k}$.

Thus, the following result has been established:

Theorem 4.8.

The $C^{A}_{n,m}(x,y;B,P)$ *satisfies the matrix differential equation*

$$
\begin{aligned}
&[(-\theta+n)(\frac{m-1}{m}\theta I-\frac{n}{m}I-A-(m-2)I)_{m-1}\\
&-myP^{m-1}(x\sqrt{mB})^{-m}(-\theta)_{m}]C^{A}_{n,m}(x,y;B,P)=0.
\end{aligned}
\tag{4.38}
$$

Finally, it is now interesting to extend the above results to new generalized forms of Gegenbauer-type matrix polynomials of two variables.

$$
C^{A}_{n,m,\nu}(x,y;B,P)=\frac{1}{n!}\Gamma^{-1}(A)\int_{0}^{\infty}e^{-Pu}u^{A+(\frac{n}{\nu}-1)I}H_{n,m,\nu}\left(x,\frac{y}{u^{\frac{m}{\nu}-1}},B\right)du \tag{4.39}
$$

where $H_{n,m,\nu}(x,y,B)$ generalized Hermite matrix polynomials is defined by Upadhyaya and Shehata [37]

$$
H_{n,m,\nu}(x,y,B)=n!\sum_{k=0}^{[\frac{1}{m}n]}\frac{(-1)^{k}y^{k}}{k!\Gamma(\frac{n-mk}{\nu}+1)}(x\sqrt{mB})^{\frac{n-mk}{\nu}} \tag{4.40}
$$

and the generating matrix function

$$
\sum_{n=0}^{\infty}\frac{t^{n}}{n!}H_{n,m,\nu}(x,y,B)=\exp(xt^{\nu}\sqrt{mB}-yt^{m}I). \tag{4.41}
$$

In fact, by noting that

$$
\begin{aligned}
&H_{n,m,\nu}(x,\alpha y,B)=\alpha^{\frac{n}{m}}H_{n,m,\nu}(\alpha^{-\frac{\nu}{m}}x,y,B),\\
&\alpha^{\frac{n}{\nu}}H_{n,m,\nu}(x,y,B)=H_{n,m,\nu}(\alpha x,\alpha^{\frac{m}{\nu}}y,B),
\end{aligned}
\tag{4.42}
$$

From (4.39) and (4.42), one gets equivalently as

$$C^{A}_{n,m,\nu}(x,y;B,P)=\frac{1}{n!}\Gamma^{-1}(A)\int_0^{\infty}e^{-Pu}u^{A-I}\ H_{n,m,\nu}(xu,yu,B)du. \tag{4.43}$$

Now, making use of equation (1.9) and the formula (4.43), we find that the Gegenbauer matrix polynomials of three matrices are defined by the following series

$$C^{A}_{n,m,\mu}(x,y;B,P)=\sum_{k=0}^{[\frac{1}{m}n]}\frac{(-1)^k y^k P^{\frac{(m-\nu)k-n}{\nu}I-A}(A)_{\frac{n-(m-\nu)k}{\nu}}}{k!\Gamma(\frac{n-mk}{\nu}+1)}(x\sqrt{mB})^{\frac{n-mk}{\nu}} \tag{4.44}$$

Theorem 4.9.

Let B *and* P *be positive stable matrices in* $C^{N\times N}$ *satisfying the conditions* $Re(\mu)>0$ *for all eigenvalues* $\mu\in\sigma(B)$, $Re(\nu)>0$ *for all eigenvalues* $\nu\in\sigma(P)$, *and let* A *be a matrix in* $C^{N\times N}$ *satisfying the condition* $(-\frac{z}{2})\notin\sigma(A)$ *for all* $z\in Z^{+}\cup\{0\}$, *such that* A ,B *and* P *are commuting matrices and* $\|B\|<\frac{1}{\sqrt{m}}$. *Then the Gegenbauer-type matrix polynomials of two variables has the following generating matrix function:*

$$\sum_{n=0}^{\infty}C^{A}_{n,m,\nu}(x,y;B,P)t^n=(P-xt^{\nu}\sqrt{mB}+yt^{m}I)^{-A} \tag{4.45}$$

where $\left\|xt^{\nu}\sqrt{mB}-yt^{m}I\right\|<\|P\|$.

Proof. Multiplying both sides of (4.43) by t^n, summing up over n, using (4.41) and them integrating over u, we have (4.45). This completes the proof.

In this paper, several new families of special matrix polynomials are introduced using integral transform method and allow the derivation of a wealth of relations involving these matrix polynomials. These results allow us to note that the use of the method of the integral representation is a fairly important tool of analysis and can be usefully extended to other families of Gegenbauer matrix polynomials which is a problem for further work.

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