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A STUDY ON COMMENSAL MORTALITY RATE OF A TYPICAL THREE SPECIES SYN-ECO-SYSTEM WITH UNLIMITED RESOURCES FOR COMMENSAL

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ABSTRACT

The present investigation is on an analytical and numerical study of a typical three species syn-eco system with mortality rate for commensal. The system comprises of a commensal (S_1), two hosts S_2 and S_3 where S_2 and S_3 both benefit S_1, without getting themselves affected either positively or adversely. Further the first species has unlimited resources. The model equations constitute a set of three first order non-linear coupled differential equations. Criteria for the asymptotic stability of all the four equilibrium states are established. Trajectories of the perturbations over the equilibrium states are illustrated. Further the global stability of the system is established with the aid of suitably constructed Liapunov's function and the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme.

Keywords: Asymptotically stable, Commensal, Equilibrium state, Host, Trajectories, Stable.

1. INTRODUCTION

Mathematical modeling has been playing an important role for the last half a century in explaining several phenomena concerned with individuals and groups of populations in nature. Significant researches in the field of theoretical ecology has been formulated by Lotka [1] and by Volterra [2]. Several mathematicians and ecologists contributed to the growth of this area of knowledge. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and so on.

The general concept of modeling has been presented in the treatises of Meyer [3], Kushing [4], Kapur [5]. Srinivas [6] studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan and Pattabhi [7] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Archana, et al. [8] and by Bhaskara and Pattabhi [9], while Ravindra [10] investigated mutualism between two species. Further Phani

[11] studied some mathematical models of ecological commensalism. The present authors Hari and Pattabhi [12], [13-16] discussed on the stability of three and four species syn-ecosystems.

The present investigation is on an analyticaland numerical study of three species (S_1, S_2, S_3) syn-eco system with mortality rate for commensal and unlimited resources for S_1. The system comprises of a commensal (S_1), two hosts S_2 and S_3 that is S_2 and S_3 both benefit S_1, without getting themselves affected either positively or adversely. Further S_2 is a commensal of S_3 and S_3 is a host of both S_1, S_2.

Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2) while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other (S_2) is called the host. Some real-life examples of commensalism are presented below.

i. Sucker fish (echeneis) gets attached to the under surface of sharks by its sucker. This provides easy transport for new feeding grounds and also food pieces falling from the sharks prey, to Echeneis.
ii. A squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.
iii. A flatworm attached to the horse crab and eating the crab's food, while the crab is not put to any disadvantage.

2. NOTATION ADOPTED

$N_i(t)$	:	The population strength of S_i at time t, $i=1,2,3$
t	:	Time instant
d_1	:	Natural death rate of S_1
a_i	:	Natural growth rate of S_i, $i=2,3$
a_{ii}	:	Self inhibition coefficients of S_i, $i=2,3$
a_{12}, a_{13}	:	Interaction coefficients of S_1 due to S_2 and S_1 due to S_3
a_{23}	:	Interaction coefficient of S_2 due to S_3
$k_i = \frac{a_i}{a_{ii}}$	:	Carrying capacities of S_i, $i=2,3$

Further the variables N_1, N_2, N_3 are non-negative and the model parameters d_1, a_2, a_3, a_{12}, $a_{22}, a_{33}, a_{13}, a_{23}, k_2, k_3$ are assumed to be non-negative constants.

3. BASIC EQUATIONS OF THE MODEL

The model equations for the three species syn ecosystem is given by the following system of first order non-linear ordinary differential equations.

(i) Equation for the first species (N_1):

$$\frac{dN_1}{dt} = N_1\left(-d_1 + a_{12}N_2 + a_{13}N_3\right) \tag{1}$$

(ii) Equation for the second species (N_2):

$$\frac{dN_2}{dt} = N_2\left(a_2 - a_{22}N_2 + a_{23}N_3\right) \tag{2}$$

(iii) Equation for the third species (N_3):

$$\frac{dN_3}{dt} = N_3\left(a_3 - a_{33}N_3\right) \tag{3}$$

4. EQUILIBRIUM STATES

The system under investigation has four equilibrium states given by $\frac{dN_i}{dt} = 0, i = 1, 2, 3$

(i) Fully washed out state.

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$$

(ii) Only the third species is washed out and the other two are not.

$$E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$$

(iii) Only the second species is washed out and the other two are not.

$$E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_2, \bar{N}_3 = 0$$

(iv) Only the first species is washed out and the other two are not.

$$E_4 : \bar{N}_1 = 0, \bar{N}_2 = k_2 + \frac{a_{23}k_3}{a_{22}}, \bar{N}_3 = k_3$$

5. STABILITY ANALYSIS

Let us consider small deviations from the steady state

that is $N_i(t) = \bar{N}_i + u_i(t), i = 1,2,3$ (4)

where $u_i(t)$ is a small perturbations in the species S_i.

The basic equations are quasi-linearized over the equilibrium state $\bar{N}=\left(\bar{N}_1,\, \bar{N}_2,\, \bar{N}_3\right)$ to obtain the equations for the perturbed state as

$$\frac{du_1}{dt}=\left(-d_1+a_{12}\bar{N}_2+a_{13}\bar{N}_3\right)u_1+\left(a_{12}\bar{N}_1\right)u_2+\left(a_{13}\bar{N}_1\right)u_3 \tag{5}$$

$$\frac{du_2}{dt}=\left(a_2-2a_{22}\bar{N}_2+a_{23}\bar{N}_3\right)u_2+\left(a_{23}\bar{N}_2\right)u_3 \tag{6}$$

$$\frac{du_3}{dt}=\left(a_3-2a_{33}\bar{N}_3\right)u_3 \tag{7}$$

The characteristic equation for the system is det $[A - \lambda I] = 0$ (8)

The equilibrium state is stable, if all the roots of the equation (8) are negative in case they are real or have negative real parts, in case they are complex.

5.1. Stability of $E_1:\bar{N}_1=0,\, \bar{N}_2=0,\, \bar{N}_3=0$

The basic equations are quasi-linearized to obtain the equations as

$$\frac{du_1}{dt}=-d_1u_1;\ \ \frac{du_2}{dt}=a_2u_2;\ \ \frac{du_3}{dt}=a_3u_3 \tag{9}$$

The characteristic equation is $\left(\lambda+d_1\right)\left(\lambda-a_2\right)\left(\lambda-a_3\right)=0$ (10)

The characteristic roots of (10) are $-d_1, a_2, a_3$. Since two of these three roots are positive. Hence the state is **unstable** and the solutions of the equations (9) are

$$u_1=u_{10}\, e^{-d_1t};\ u_2=u_{20}\, e^{a_2t};\ u_3=u_{30}\, e^{a_3t} \tag{11}$$

where u_{10}, u_{20}, u_{30} are the initial values of u_1, u_2, u_3 respectively.

Trajectories of perturbations

The trajectories in u_1-u_2 and u_2-u_3 planes are

$$\left(\frac{u_1}{u_{10}}\right)^{-\frac{1}{d_1}}=\left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}}=\left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_3}}$$

5.2. Stability of $E_2:\bar{N}_1=0,\, \bar{N}_2=0,\, \bar{N}_3=k_3$

In this state, the basic equations can be quasi-linearized, we get

$$\frac{du_1}{dt}=(a_{13}k_3-d_1)u_1;\ \frac{du_2}{dt}=(a_2+a_{23}k_3)u_2;\ \frac{du_3}{dt}=-a_3u_3 \tag{12}$$

The characteristic roots are $a_{13}k_3-d_1$, $a_2+a_{23}k_3$ and $-a_3$. Since one of these three roots is positive, hence the state is **unstable** and the solutions are

Case (i): When $a_{13}k_3<d_1$

In this case, the solutions are

$$u_1=u_{10}e^{-(a_{13}k_3+d_1)\,t};u_2=u_{20}\,e^{(a_2+a_{23}k_3)\,t};u_3=u_{30}e^{-a_3t} \tag{13}$$

Case (ii): When $a_{13}k_3>d_1$

In this case, the solutions are

$$u_1=u_{10}e^{(a_{13}k_3+d_1)\,t};u_2=u_{20}\,e^{(a_2+a_{23}k_3)\,t};u_3=u_{30}e^{-a_3t} \tag{14}$$

Case (iii): When $a_{13}k_3=d_1$

In this case, the solutions are

$$u_1=u_{10};\ u_2=u_{20}\,e^{(a_2+a_{23}k_3)\,t};u_3=u_{30}e^{-a_3t} \tag{15}$$

Trajectories of perturbations

The trajectories in the u_1-u_2 and u_2-u_3 planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{d_1-a_{13}k_3}}=\left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2+a_{23}k_3}}=\left(\frac{u_3}{u_{30}}\right)^{-\frac{1}{a_3}}$$

5.3. Stability of $E_3:\bar{N}_1=0,\bar{N}_2=k_2,\bar{N}_3=0$

The basic equations can be quasi-linearized,
We get

$$\frac{du_1}{dt}=(a_{12}k_2-d_1)u_1;\ \frac{du_2}{dt}=-a_2u_2+a_{23}k_2u_3;\ \frac{du_3}{dt}=a_3u_3 \tag{16}$$

The characteristic roots are $a_{12}k_2-d_1$, $-a_2$ and a_3. Since one of these three roots is positive, hence the state is **unstable**. The equations (16) yield the solutions,

Case (i): When $a_{12}k_2<d_1$

In this case, the solutions are

$$u_1 = u_{10} e^{-(a_{12}k_2 + d_1)\,t}; u_2 = (1 - u_{30}\alpha) u_{20} e^{-a_2 t} + (u_{20}\alpha) u_{30} e^{a_3 t}; u_3 = u_{30} e^{a_3 t} \tag{17}$$

where $\alpha = \dfrac{a_{23}k_2}{u_{20}(a_2 + a_3)}$ (18)

Case (ii): When $a_{12}k_2 > d_1$

In this case, the solutions are

$$u_1 = u_{10} e^{(a_{12}k_2 + d_1)\,t}; u_2 = (1 - u_{30}\alpha) u_{20} e^{-a_2 t} + (u_{20}\alpha) u_{30} e^{a_3 t}; u_3 = u_{30} e^{a_3 t} \tag{19}$$

Case (iii): When $a_{12}k_2 = d_1$

In this case, the solutions are

$$u_1 = u_{10};\ u_2 = (1 - u_{30}\alpha) u_{20} e^{-a_2 t} + (u_{20}\alpha) u_{30} e^{a_3 t}; u_3 = u_{30} e^{a_3 t} \tag{20}$$

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are

$$\frac{u_2}{u_{20}} = (1 - u_{30}\alpha)\left(\frac{u_1}{u_{10}}\right)^{\frac{-a_2}{a_1 + a_{12}k_2}} + u_{30}\alpha\left(\frac{u_1}{u_{10}}\right)^{\frac{a_3}{a_1 + a_{12}k_2}};\quad \frac{u_2}{u_{20}} = (1 - u_{30}\alpha)\left(\frac{u_3}{u_{30}}\right)^{\frac{-a_2}{a_3}} + u_3\alpha$$

5.4. Stability of $E_4 : \bar{N}_1 = 0, \bar{N}_2 = k_2 + \dfrac{a_{23}k_3}{a_{22}}, \bar{N}_3 = k_3$

In this state, the basic equations can be quasi-linearized, we have

$$\frac{du_1}{dt} = (\alpha_1 - d_1) u_1;\ \frac{du_2}{dt} = -(a_2 + a_{23}) u_2 + \frac{a_{23}}{a_{22}}(a_2 + a_{23}) u_3;\ \frac{du_3}{dt} = -a_3 u_3 \tag{21}$$

where $\alpha_1 = a_{13}k_3 + \dfrac{a_{12}}{a_{22}}(a_2 + a_{23}k_3) > 0$ (22)

The characteristic roots are $\alpha_1 - d_1$, $-(a_2 + a_{23}k_3)$ and $-a_3$.

Case (i): When $\alpha_1 < d_1$

In case all the three roots are negative, hence the state is **stable**. The equations (21) yield the solutions.

$$u_1 = u_{10} e^{-(\alpha_1 + d_1)t}; u_2 = (1 - u_{30}\beta) u_{20} e^{-(a_2 + a_{23}k_3)t} + (u_{20}\beta) u_{30} e^{-a_3 t}; u_3 = u_{30} e^{-a_3 t} \tag{23}$$

where $\beta = \dfrac{a_{23}(a_2 + a_{23}k_3)}{a_{22}u_{20}(a_2 + a_{23}k_3 - a_3)}$ (24)

Case (ii): When $\alpha_1 = d_1$

In case the state is **neutrally stable**. The equations (21) yield the solutions.

$$u_1 = u_{10};\, u_2 = \left(1 - u_{30}\beta\right) u_{20} e^{-(a_2 + a_{23}k_3)t} + \left(u_{20}\beta\right) u_{30} e^{-a_3 t};\, u_3 = u_{30} e^{-a_3 t} \tag{25}$$

Case (iii): When $\alpha_1 > d_1$

In case the state is **unstable** and the equations (21) yield the solutions.

$$u_1 = u_{10} e^{(\alpha_1 + d_1)t};\, u_2 = \left(1 - u_{30}\beta\right) u_{20} e^{-(a_2 + a_{23}k_3)t} + \left(u_{20}\beta\right) u_{30} e^{-a_3 t};\, u_3 = u_{30} e^{-a_3 t} \tag{26}$$

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\frac{u_2}{u_{20}} = \left(1 - u_{30}\beta\right)\left(\frac{u_1}{u_{10}}\right)^{\frac{a_2 + a_{23}k_3}{d_1 - \alpha_1}} + u_{30}\beta\left(\frac{u_1}{u_{10}}\right)^{\frac{a_3}{d_1 - \alpha_1}};\; \frac{u_2}{u_{20}} = \left(1 - \beta u_{30}\right)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_2 + a_{23}k_3}{a_3}} + \beta u_3$$

6. LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY

In section 5 we discussed the local stability of all four equilibrium states. From which only one state $E_4\left(0, \bar{N}_2, \bar{N}_3\right)$ is **stable** and rest of them are **unstable**. We now examine the global stability of dynamical system (1), (2) and (3) at this state by suitable Liapunov's function.

Theorem: *The equilibrium state* $E_4\left(0,\, k_2 + \dfrac{a_{23}k_3}{a_{22}},\, k_3\right)$ *is globally asymptotically stable.*

Proof: Let us consider the following Liapunov's function

$$L\left(N_2, N_3\right) = N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right) + l_1\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right] \tag{27}$$

where l_1 is a suitable constant to be determined as in the subsequent steps.

Now, the time derivative of L, along with solutions of (2) and (3) can be written as

$$\frac{dL}{dt} = \left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + l_1\left(\frac{N_3 - \bar{N}_3}{N_3}\right)\frac{dN_3}{dt} \tag{28}$$

$$= -a_{22}\left(N_2-\bar{N}_2\right)^2 + a_{23}\left(N_2-\bar{N}_2\right)\left(N_3-\bar{N}_3\right) + l_1\left[-a_{33}\left(N_3-\bar{N}_3\right)^2\right]$$

$$\frac{dL}{dt} = -\left[\sqrt{a_{22}}\left(N_2-\bar{N}_2\right)+\sqrt{l_1 a_{33}}\left(N_3-\bar{N}_3\right)\right]^2 - \left(2\sqrt{l_1 a_{22} a_{33}} - a_{23}\right)\left(N_2-\bar{N}_2\right)\left(N_3-\bar{N}_3\right) \quad (29)$$

The positive constant l_1 as so chosen that, the coefficient of $\left(N_2-\bar{N}_2\right)\left(N_3-\bar{N}_3\right)$ in (29) vanish.

Then we have $l_1 = \dfrac{a_{23}^2}{4a_{22}a_{33}} > 0$ and, with this choice of the constant l_1

$$\frac{dL}{dt} = -\left[\sqrt{a_{22}}\left(N_2-\bar{N}_2\right) - \frac{a_{23}}{2\sqrt{a_{22}}}\left(N_3-\bar{N}_3\right)\right]^2 < 0 \quad (30)$$

Hence, the steady state is **globally asymptotically stable.**

7. NUMERICAL APPROACH

The numerical solution of the growth rate equations computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. For this, MAT LAB has been used and the results are illustrated in Figures 1 to 3.

Consider the parameters,

$$d_1 = 2.49, a_{12} = 0.72, a_{13} = 0.288, a_2 = 1.188, a_{22} = 4.536, a_{23} = 2.034, a_3 = 0.594, a_{33} = 0.342$$

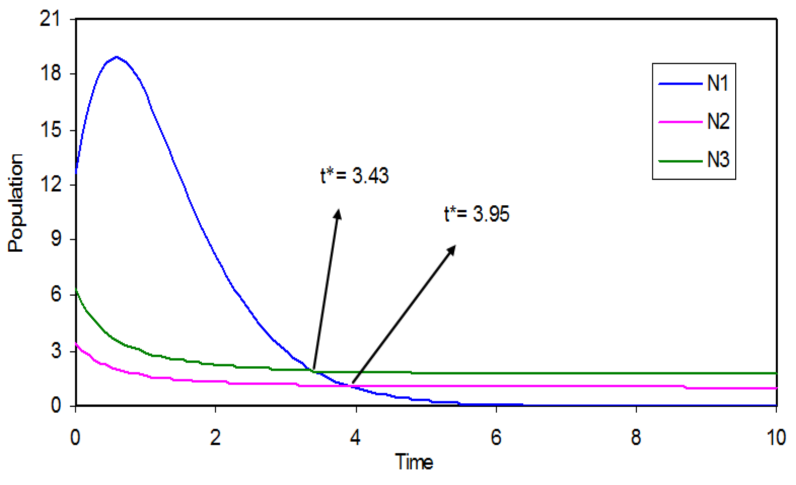

Figure-1. Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 12.69, N_{20} = 3.438, N_{30} = 6.408$

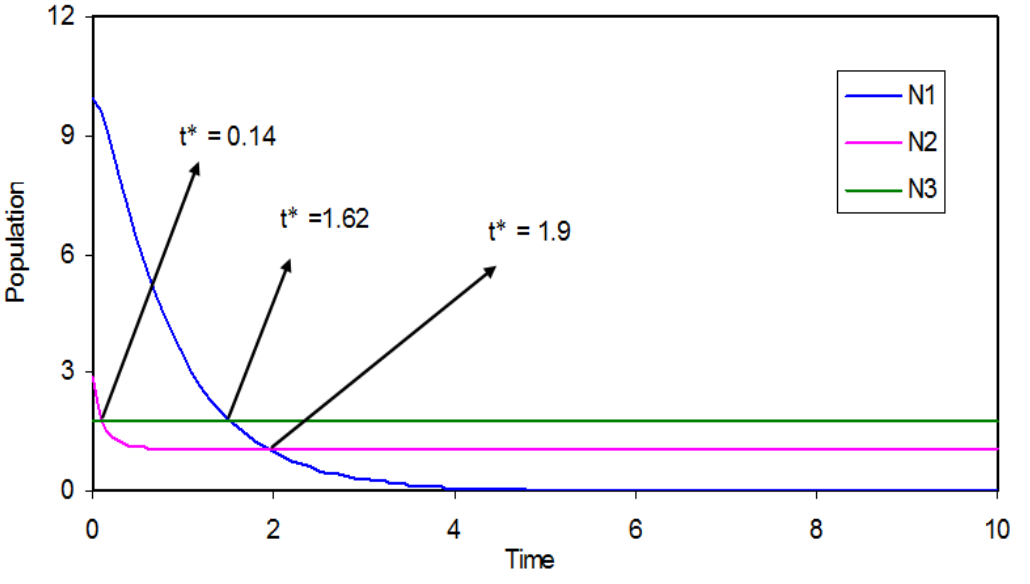

Figure-2. Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 9.936, N_{20} = 2.916, N_{30} = 1.746$

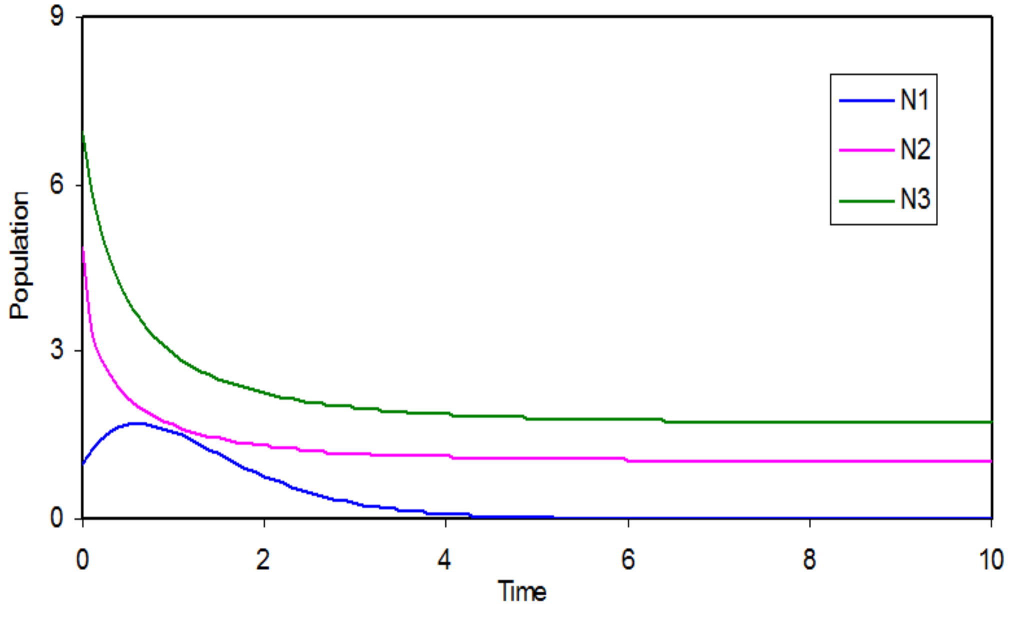

Figure-3. Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 1, N_{20} = 4.842, N_{30} = 6.93$

8. CONCLUSION

The present paper deals with an investigation on the stability of a typical three species syn eco-system with mortality rate for the commensal. In this paper we established all possible equilibrium states. It is conclude that, in all four equilibrium states, only one state E_4 is conditionally stable. Further the global stability is established with the help of suitable Liapunov's function and the growth rates of the species are numerically estimated using Runge-Kutta fourth order method.

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