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ON HOMOGENEOUS CUBIC EQUATION WITH FOURUNKNOWNS $x^3 + y^3 = 21zw^2$

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ABSTRACT
The homogeneous cubic equation with four unknowns represented by the Diophantine equation $x^3 + y^3 = 21zw^2$ *is analyzed for its patterns of non – zero distinct integer solutions. A few interesting properties between the solutions and special numbers, namely, Polygonal number, Pyramidal number, Centered polygonal number, Stella octangular number and Octahedral number are presented.*

Keywords: Homogeneous cubic, Cubic equation with four unknowns, Integral solutions, Cubic Diophantine equation, Third degree equation, Special numbers.

Notations Used

- Polygonal number of rank n with size m.

$$t_{m,n} = n[1 + \frac{(n-1)(m-2)}{2}]$$

- Pyramidal number of rank n with size m.

$$p_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)]$$

- Centered polygonal number of rank n with size m.

$$ct_{m,n} = \frac{mn(n+1)+2}{2}$$

- Stella octangular number of rank n

$$SO_n = n(2n^2 - 1)$$

- Octahedral number of rank n

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$$OH_n = \frac{1}{3}n(2n^2+1)$$

Contribution/ Originality

This study contributes in the existing literature different approaches of determining non-zero distinct integer solutions to the homogeneous equation of degree three with 4 unknowns given by $x^3+y^3=21zw^2$

1. INTRODUCTION

Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity can be seen as in [1-3].The Diophantine equations offer an unlimited field for research due to their variety. The problem of finding all integer solutions of a Diophantine equation with three or more variables and degree at least three, in general presents a good deal of difficulties. Cubic equation in three variables falls into the theory but is still an important topic of current research [4-6]. Equations with more than three variables and degree at least three are known very little.

In particular, one may refer [7-18] for cubic equations with four unknowns. This research concerns with yet another interesting equation $x^3+y^3=21zw^2$ representing the homogeneous cubic equation with four unknowns for determining its infinitely many non-zero integer points. Also a few interesting properties are presented.

2. SOME INTERESTING PATTERNS

The homogeneous cubic Diophantine equation with four unknowns to be solved is given by

$$x^3+y^3=21zw^2 \quad (1)$$

which is written as $(x+y)(x^2-xy+y^2)=21zw^2$ (2)

Suppose $z=x+y$ (3)

Substitute (3) into (2), it reduces to the quadratic equation $(x^2-xy+y^2)=21w^2$ (4)

Let $x=u+v, y=u-v$ (5)

where u and v are non-zero distinct arbitrary integers.

Substituting (5) in (4), it gives $u^2+3v^2=21w^2$ (6)

Equation (6) is solved through different approaches and the different patterns of solutions of (1) obtained are presented below.

2.1 Pattern-1

Assume $w = a^2 + 3b^2 = (a + i\sqrt{3}b)(a - i\sqrt{3}b)$ (7)

Write 21 as $21 = (3 + i2\sqrt{3})(3 - i2\sqrt{3})$ (8)

Using (7) and (8) in (6), it is written as

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (3 + i2\sqrt{3})(3 - i2\sqrt{3})(a + i\sqrt{3}b)^2(a - i\sqrt{3}b)^2 \quad (9)$$

Suppose that following system of equations are derived from (9)

$$(u + i\sqrt{3}v) = (3 + i2\sqrt{3})(a + i\sqrt{3}b)^2$$

$$(u - i\sqrt{3}v) = (3 - i2\sqrt{3})(a - i\sqrt{3}b)^2$$

Equating the real and imaginary parts in either of the above two equations, we get

$$u = 3a^2 - 9b^2 - 12ab$$

$$v = 2a^2 - 6b^2 + 6ab$$

Hence, in view of (3) and (5), we have

$$\left\{\begin{aligned} x &= x(a,b) = 5a^2 - 15b^2 - 6ab \\ y &= y(a,b) = a^2 - 3b^2 - 18ab \\ z &= z(a,b) = 6a^2 - 18b^2 - 24ab \end{aligned}\right\} \quad (10)$$

Thus (7) and (10) represent non-zero distinct integer solutions for (1).

Properties of pattern-1: It is easy to infer following properties from (10)

- $x(n,2n^2-1) - 5y(n,2n^2-1) = 6y(n,2n^2-1) - z(n,2n^2-1) = 84n(2n^2-1) = 84SO_n$
- $z(n,n(n+1)) - x(n,n(n+1)) - w(n,n(n+1)) - 9p_n^5 = 6t_{4,n}$
- $x(n,2n^2+1) - 5y(n,2n^2+1) = 84(n,2n^2+1) = 252OH_n$
- $x(n,n) - z(n,n) + w(n,n) - 8t_{4,n} = 16n^2$, a perfect square
- $x(n,n) - z(n,n) + w(n,n) = 24n^2$, a nasty number.

Notes of pattern-1:

Instead of (8), write 21 as $21=\frac{(3+i5\sqrt{3})-(3-i5\sqrt{3})}{4}$

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

$$x=x(a,b)=4a^2-12b^2-12ab$$
$$y=y(a,b)=-a^2+3b^2-18ab$$
$$z=z(a,b)=3a^2-9b^2-30ab$$
$$w=w(a,b)=a^2+3b^2$$

2.2 Pattern-2

Equation (6) can be written as $u^2+3v^2=21w^2*1$ (11)

Write 1 as $1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$ (12)

Using (7), (8) and (12) in (11), it is written as

$$(u+i\sqrt{3}v)(u-i\sqrt{3}v)=(3+i2\sqrt{3})(3-i2\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2\left(\frac{1+i\sqrt{3}}{2}\right)\left(\frac{1-i\sqrt{3}}{2}\right)$$

Consider $(u+i\sqrt{3}v)=(3+i2\sqrt{3})(a+i\sqrt{3}b)^2\left(\frac{1+i\sqrt{3}}{2}\right)$

Equating real and imaginary parts, we have

$$u=\frac{1}{2}\left[-3a^2+9b^2-30ab\right]$$
$$v=\frac{1}{2}\left[5a^2-15b^2-6ab\right]$$

Substituting the above value of u and v in (3) and (5), we obtain

$$\left\{\begin{array}{l}x=x(a,b)=a^2-3b^2-18ab\\ y=y(a,b)=-4a^2+12b^2-12ab\\ z=z(a,b)=-3a^2+9b^2-30ab\end{array}\right\} \quad (12a)$$

Thus (7) and (12a) represent non-zero distinct integer solutions for (1).

Properties of pattern-2: It is easy to infer following properties from (12a)

- $x(n,-n)+w(n,-n)=20n^2=20t_{4,n}$

- $4x(n,n+1)+y(n,n+1)=-84n(n+1)=3x(n,n+1)+z(n,n+1)$
- $3w(n,n)-z(n,n)=36n^2$ is a perfect square
- $3\{x(n,2n^2-1)+w(n,2n^2-1)-18SO_n\}=6n^2$, a nasty number
- $21^2\{17x(-n,n^2)+y(-n,n^2)+z(-n,n^2)\}=(42n)^3$, a cubical integer.

2.3 Pattern-3

Instead of (12), write 1 as $1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}$

Repeating the above process as in pattern-2, the non-zero distinct integral solutions of (1) are found to be

$$x=x(a,b)=-a^2+3b^2-18ab$$
$$y=y(a,b)=-5a^2+15b^2-6ab$$
$$z=z(a,b)=-6a^2+18b^2-24ab$$
$$w=w(a,b)=a^2+3b^2$$

Properties of pattern-3: It is easy to infer following properties from above equations

- $5x(n^2,n+1)-y(n^2,n+1)=-84(n^2,n+1)\equiv 0(\mathrm{mod}\,84)$
- $3x(n,n)-y(n,n)+165t_{4,n}=81n^2$, a perfect square
- $w(n,n)+x(n,n)$ + perfect square = $n^2=t_{4,n}$
- $6x(n,2n^2+1)-z(n,2n^2+1)=252OH_n$

2.4 Pattern-4

One may write (6) as $(u^2-9w^2)=3(4w^2-v^2)$ (13)

Write (13) in the form of ratio as

$$\left(\frac{u+3w}{3(2w-v)}\right)=\left(\frac{2w+v}{u-3w}\right)=\left(\frac{a}{b}\right), b\neq 0$$

Which is equivalent to the system of double equations

$$ub + 3va + w(3b - 6a) = 0$$
$$-ua + vb + w(2b + 3a) = 0$$

Applying the method of cross-multiplication, we have

$$\begin{aligned} u &= 9a^2 - 3b^2 + 12ab \\ v &= 6a^2 - 2b^2 - 6ab \end{aligned} \qquad (14)$$

$$w = 3a^2 + b^2 \qquad (15)$$

Hence, in view of (3) and (5), the corresponding values of x, y and z are given by

$$\left\{\begin{array}{l} x = x(a,b) = 15a^2 - 5b^2 + 6ab \\ y = y(a,b) = 3a^2 - b^2 + 18ab \\ z = z(a,b) = 18a^2 - 6b^2 + 24ab \end{array}\right\} \qquad (15a)$$

Thus (15) and (15a) represent non-zero distinct integer solutions for (1).

Properties of pattern-4: It is easy to infer following properties from (15a)

- $5y(n,2n^2-1) - x(n,2n^2-1) - 2SO_n = 82(n,2n^2-1) \equiv 0(\bmod 82)$
- $x(n,n+1) + y(n,n+1) - z(n,n+1) = 0$
- $y(n,19n^2-13) + w(n,19n^2-13) - 108CP_{19,n} - t_{14,n} = 5n \equiv 0(\bmod 5)$
- Each of the following represents a perfect square
  - $6\{y(n,19n^2-13) + w(n,19n^2-13) - 108CP_{19,n}\} = 36n^2$
  - $21\{6y(n,n) - z(n,n)\} = (42n)^2$

2.5 Pattern-5

Equation (6) can be written as $3v^2 = 21w^2 - u^2$ (16)

Write $v = 21a^2 - b^2$ (17)

Write 3 as $3 = \dfrac{(\sqrt{21}+3)(\sqrt{21}-3)}{4}$ (17a)

Substituting (17) and (17a) in (16), we get

$$(\sqrt{21}w+u)(\sqrt{21}w-u)=(\sqrt{21}a+b)^2(\sqrt{21}a-b)^2\left(\frac{\sqrt{21}+3}{2}\right)\left(\frac{\sqrt{21}-3}{2}\right)$$

Consider

$$(\sqrt{21}w+u)=(\sqrt{21}a+b)^2\left(\frac{\sqrt{21}+3}{2}\right)$$

Equating rational and irrational parts, we have

$$u=\frac{1}{2}\left[63a^2+3b^2+42ab\right] \quad \text{and} \quad w=\frac{1}{2}\left[21a^2+b^2+6ab\right] \qquad (18)$$

Replacing a by 2A and b by 2B in (18), and using (3), (5),(17) and (17a), we have

$$x=x(A,B)=210A^2+2B^2+24AB$$
$$y=y(A,B)=42A^2+10B^2+84AB$$
$$z=z(A,B)=252A^2+12B^2+168AB$$
$$w=w(A,B)=42A^2+2B^2+12AB$$

Properties of pattern-5:

- $6[y(n,n^2)-w(n,n^2)]-5y(n,n^2)+x(n,n^2)=96n^3=96CP_{6,n}$
- $z(n,n+1)-6w(n,n+1)=96n(n+1)=6[y(n,n+1)-w(n,n+1)]-[5y(n,n+1)-x(n,n+1)]$
- $y(n,n)-w(n,n)=80n^2\equiv 0(\mathrm{mod}\,80)$
- $y(n,n)-w(n,n)+t_{4,A}=81n^2$, a perfect square
- $z(n,n)-6w(n,n)=96n^2$ =, a Nasty number

3. CONCLUSION

In this paper, we have illustrated different ways of obtaining non-zero distinct integer solutions to the homogeneous cubic equation with four unknowns given by $x^3+y^3=21zw^2$. As the Diophantine equations are rich in variety, one may search for the integral solutions of other forms of cubic Diophantine equations along with their corresponding properties.

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REFERENCES

[1] L. E. Dickson, *History of theory of numbers, diophantine analysis* vol. 2. New York: Dover, 2005.

[2] L. J. Mordell, *Diophantine equations.* New York: Academic Press, 1969.

[3] R. D. Carmichael, *The theory of numbers and diophantine analysis.* New York: Dover, 1959.

[4] M. A. Gopalan, S. Manju, and N. Vanitha, "On ternary cubic diophantine equation $(x^2+y^2)=2z^3$," *Advances in Theoretica and Applied Mathematics*, vol. 1, pp. 227-231, 2006.

[5] M. A. Gopalan, S. Manju, and N. Vanitha, "Ternary cubic diophantine equation $2^{2a-1}(x^2+y^2)=z^3$," *Acta Ciencia Indica*, vol. 34M, pp. 1135-1137, 2008.

[6] M. A. Gopalan and G. Sangeetha, "On the ternary cubic diophantine equation $y^2=Dx^2+z^3$," *Archimedeas Journal of Mathematics*, vol. 1, pp. 7-14, 2011.

[7] M. A. Gopalan and S. Premalatha, "Integral solutions of $(x+y)(xy+w^2)=2(k^2+1)z^3$," *Bulletin of Pure and Applied Sciences*, vol. 29E, pp. 197-202, 2009.

[8] M. A. Gopalan and V. Pandi Chelvi, "Remarkable solutions on the cubic equation with four unknows $x^3+y^3+z^3=28(x+y+z)w^2$," *Antarctica J. Math*, vol. 4, pp. 393-401, 2010.

[9] M. A. Gopalan and B. Sivagami, "Integral solutions of homogeneous cubic equation with four unknows $x^3+y^3+z^3=3xyz+2(x+y)w^3$," *Impact.J.Sci.Tech.*, vol. 4, pp. 53-60, 2010.

[10] M. A. Gopalan and S. Premalatha, "On the cubic diophantine equation with four unknows $(x-y)(xy-w^2)=2(n^2+2n)z^3$," *International Journal of Mnathematical Sciences*, vol. 9, pp. 171-175, 2010.

[11] M. A. Gopalan and J. Kaliga Rani, "Integral solutions of $x^3+y^3+(x+y)xy=z^3+w^3+(z+w)zw$," *Bulletin of Pure and Applied Sciences*, vol. 29E, pp. 169-173, 2010.

[12] M. A. Gopalan and S. Premalatha, "Integral solutions of $(x+y)(xy+w^2)=2(k+1)z^3$," *The Global Journal of Applied Mathematics and Mathematical Sciences*, vol. 3, pp. 51-55, 2010.

[13] M. A. Gopalan, S. Vidhyalakshmi, and S. Mallika, "Observation on the cubic equation with four unknows $xy+2z^2=w^3$" *The Global Journal of Mathematics and Mathematical Sciences*, vol. 2, pp. 69-74, 2012.

[14] M. A. Gopalan, S. Vidhyalakshmi, and S. Mallika, "Observation on the cubic equation with four unknows $2(x^3+y^3)=z^3+w^2(x+y)$," *IJAMP*, vol. 4, pp. 103-107, 2012.

[15] M. A. Gopalan, S. Vidhyalakshmi, and G. Sumathi, "On the homogeneous cubic equation with four unknows $x^3+y^3=14z^3-3w^2(x+y)$," *Discovery*, vol. 2, pp. 17-19, 2012.

[16] M. A. Gopalan and K. Geetha, "Observation on the cubic equation with four unknows, $x^3+y^3+xy(x+y)=z^3+2(x+y)w^2$" *International Journal of Pure and Applied Mathematics Sciences*, vol. 6, pp. 25-30, 2013.

[17] M. A. Gopalan, S. Vidhyalaksmi, and N. Thiruniraiselvi, "On homogeneous cubic equation with four unknowns $(x+y+z)^3=z(xy+31w^2)$," *Cayley J.Math.*, vol. 2, pp. 163-168, 2013.

[18] M. A. Gopalan, S. Vidhyalaksmi, and A. Kavitha, "Observations on the homogeneous cubic equation with four unknowns $(x+y)(2x^2+2y^2-3xy)=(k^2+7)zw^2$," *Bessel J.Math.*, vol. 4, pp. 1-6, 2014.