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SOME MODIFIED LINEAR REGRESSION TYPE RATIO ESTIMATORS FOR ESTIMATION OF POPULATION MEAN USING KNOWN PARAMETERS OF AN AUXILIARY VARIABLE

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ABSTRACT

The present paper deals with modified linear regression type ratio estimators for estimation of population mean of the study variable when the Kurtosis, Skewness, population correlation coefficient and quartiles of the auxiliary variable are known. The bias and the mean squared error of the proposed estimators are derived and are compared with that of simple random sampling without replacement (SRSWOR) sample mean, the usual ratio estimator and the existing modified linear regression type ratio estimators. As a result, we have derived the conditions for which the proposed estimators perform better than the other existing estimators. Further the performance of the proposed estimators with that of the existing estimators are assessed for a natural population. From the numerical study it is observed that the proposed modified ratio estimators perform better than the existing estimators.

Keywords: First quartile, Inter-quartile range, Simple random sampling, Semi-quartile average, Semi-quartile range.

1. INTRODUCTION

Consider a finite population $U = \{U_1, U_2, \ldots, U_N\}$ of N distinct and identifiable units. Let Y be a study variable with value Y_i measured on $U_i, i = 1, 2, 3, \ldots, N$ giving a vector $Y = \{Y_1, Y_2, \ldots, Y_N\}$. The problem is to estimate the population mean $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ on the basis of a random sample selected from the population U. The SRSWOR sample mean is the simplest estimator for estimating the population mean. If an auxiliary variable X, closely related to the study variable Y is available then one can improve the performance of the estimator of the study variable by using the known values of the population parameters of the auxiliary variable. That is, when the population parameters of the auxiliary variable X such as population mean, coefficient of variation, kurtosis, skewness, median are known, a number of estimators such as ratio, product and linear regression estimators and their modifications are suggested in literature and are performing better than the SRSWOR sample mean under certain conditions. Among these estimators the ratio estimator and its modifications are widely attracted many researchers for the estimation of the mean of the study variable (see for example Murthy [1], Singh and Chaudhary [2], Al-Omari, et al. [3], Kadilar and Cingi [4, 5], Yan and Tian [6], Subramani [7] and Cingi and Kadilar [8]).

Before discussing further about the existing modified ratio estimators and the proposed modified ratio estimators, the notations to be used in this paper are described below:

- N – Population size
- n – Sample size
- $f = n/N$, Sampling fraction
- Y – Study variable
- X – Auxiliary variable
- $\bar{X}, \bar{Y}$ – Population means
- x, y - Sample totals
- $\bar{x}, \bar{y}$ – Sample means
- S_x, S_y – Population standard deviations
- S_{xy} – Population covariance between X and Y
- C_x, C_y – Coefficient of variations
- ρ – Coefficient of correlation between X and Y
- $\mu_r = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^r$, r^{th} order central moment
- $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$, Coefficient of skewness of the auxiliary variable
- $\beta_2 = \frac{\mu_4}{\mu_2^2}$, Coefficient of kurtosis of the auxiliary variable
- $b = \beta = \frac{S_{xy}}{S_x^2}$, Regression coefficient of Y on X
- Q_1 –First (lower) quartile of auxiliary variable
- Q_3 –Third (upper) quartile of auxiliary variable
- $Q_r = (Q_3 - Q_1)$ –Inter-quartile range of auxiliary variable
- $Q_d = \frac{(Q_3 - Q_1)}{2}$ –Semi-quartile range of auxiliary variable
- $Q_a = \frac{(Q_3 + Q_1)}{2}$ –Semi-quartile average of auxiliary variable
- $B(.)$ – Bias of the estimator
- $MSE(.)$ – Mean squared error of the estimator

- $\hat{\bar{Y}}_i(\hat{\bar{Y}}_{pj})$ – i^{th}Existing (j^{th}proposed) modified ratio estimator of $\bar{Y}$

In the case of simple random sampling without replacement (SRSWOR), the sample mean $\bar{y}_{srs}$is used to estimate population mean$\bar{Y}$which is an unbiased estimator and its variance is given below:

$$V(\bar{y}_{srs}) = \frac{(1-f)}{n} S_y^2 \qquad (1)$$

The ratio estimator for estimating the population mean $\bar{Y}$ of the study variable Y is defined as

$$\hat{\bar{Y}}_R = \frac{\bar{y}}{\bar{x}} \bar{X} \qquad (2)$$

The bias and mean squared error of $\hat{\bar{Y}}_R$ to the first order of approximation are given below:

$$B\left(\hat{\bar{Y}}_R\right) = \frac{(1-f)}{n} \bar{Y}\left(C_x^2 - \rho C_x C_y\right) \qquad (3)$$

$$MSE\left(\hat{\bar{Y}}_R\right) = \frac{(1-f)}{n} \bar{Y}^2\left(C_y^2 + C_x^2 - 2\rho C_x C_y\right) \qquad (4)$$

The usual linear regression estimator and its variance are given below:

$$\hat{\bar{Y}}_{lr} = \bar{y} + \beta(\bar{X} - \bar{x}) \qquad (5)$$

$$V\left(\hat{\bar{Y}}_{lr}\right) = \frac{(1-f)}{n} S_y^2(1 - \rho^2) \qquad (6)$$

The ratio and the linear regression estimators are used for improving the precision of estimate of the population mean based on SRSWOR when there exist an auxiliary variable X and positively correlated with$Y$. It has been established; in general that the linear regression estimator is more efficient than the ratio estimator whenever the regression line of the study variable on the auxiliary variable does not passes through the neighbourhood of the origin. Thus, the linear regression estimator is more precise than the ratio estimator unless $\beta = R$ (See page 196 in Cochran [9]).

Kadilar and Cingi [4] have suggested a class of modified ratio estimators in the form of linear regression estimator as given below:

$$\hat{\bar{Y}}_1 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\bar{X}}{\bar{x}}\right] \qquad (7)$$

Kadilar and Cingi [4] have shown that the modified linear regression type ratio estimator given in (7) perform well compare to the usual ratio estimator under certain conditions. Further estimators are proposed by modifying the estimator given in (7) by using the known Coefficient of Variation, Coefficient of Kurtosis, Coefficient of Skewness etc. together with some other modified ratio estimators in Kadilar and Cingi [4], Kadilar and Cingi [5] and Yan and Tian [6]. The list of existing modified linear regression type ratio estimators together with the constant, the bias and the mean squared error are given below in Table 1:

Table-1. Existing modified ratio estimators (Class 1) together with the constant, bias and mean squared error

Estimators	Constant R_i	Bias-B(.)	MSE(.)
$\hat{\bar{Y}}_1 = [\bar{y} + b(\bar{X} - \bar{x})]\frac{\bar{X}}{\bar{x}}$ Kadilar and Cingi [4]	$R_1 = \frac{\bar{Y}}{\bar{X}}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_1^2$	$\frac{(1-f)}{n}\left(R_1^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_2 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\bar{X} + C_x}{\bar{x} + C_x}\right]$ Kadilar and Cingi [4]	$R_2 = \frac{\bar{Y}}{\bar{X} + C_x}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_2^2$	$\frac{(1-f)}{n}\left(R_2^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_3 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\bar{X} + \beta_2}{\bar{x} + \beta_2}\right]$ Kadilar and Cingi [4]	$R_3 = \frac{\bar{Y}}{\bar{X} + \beta_2}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_3^2$	$\frac{(1-f)}{n}\left(R_3^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_4 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\beta_2\bar{X} + C_x}{\beta_2\bar{x} + C_x}\right]$ Kadilar and Cingi [4]	$R_4 = \frac{\beta_2\bar{Y}}{\beta_2\bar{X} + C_x}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_4^2$	$\frac{(1-f)}{n}\left(R_4^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_5 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{C_x\bar{X} + \beta_2}{C_x\bar{x} + \beta_2}\right]$ Kadilar and Cingi [4]	$R_5 = \frac{C_x\bar{Y}}{C_x\bar{X} + \beta_2}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_5^2$	$\frac{(1-f)}{n}\left(R_5^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_6 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\bar{X} + \beta_1}{\bar{x} + \beta_1}\right]$ Yan and Tian [6]	$R_6 = \frac{\bar{Y}}{\bar{X} + \beta_1}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_6^2$	$\frac{(1-f)}{n}\left(R_6^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_7 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\beta_1\bar{X} + \beta_2}{\beta_1\bar{x} + \beta_2}\right]$ Yan and Tian [6]	$R_7 = \frac{\beta_1\bar{Y}}{\beta_1\bar{X} + \beta_2}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_7^2$	$\frac{(1-f)}{n}\left(R_7^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_8 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\bar{X} + \rho}{\bar{x} + \rho}\right]$ Kadilar and Cingi [5]	$R_8 = \frac{\bar{Y}}{\bar{X} + \rho}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_8^2$	$\frac{(1-f)}{n}\left(R_8^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_9 = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{C_x\bar{X} + \rho}{C_x\bar{x} + \rho}\right]$ Kadilar and Cingi [5]	$R_9 = \frac{C_x\bar{Y}}{C_x\bar{X} + \rho}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_9^2$	$\frac{(1-f)}{n}\left(R_9^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_{10} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\rho\bar{X} + C_x}{\rho\bar{x} + C_x}\right]$ Kadilar and Cingi [5]	$R_{10} = \frac{\rho\bar{Y}}{\rho\bar{X} + C_x}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{10}^2$	$\frac{(1-f)}{n}\left(R_{10}^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_{11} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\beta_2\bar{X} + \rho}{\beta_2\bar{x} + \rho}\right]$ Kadilar and Cingi [5]	$R_{11} = \frac{\beta_2\bar{Y}}{\beta_2\bar{X} + \rho}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{11}^2$	$\frac{(1-f)}{n}\left(R_{11}^2S_x^2 + S_y^2(1-\rho^2)\right)$
$\hat{\bar{Y}}_{12} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\rho\bar{X} + \beta_2}{\rho\bar{x} + \beta_2}\right]$ Kadilar and Cingi [5]	$R_{12} = \frac{\rho\bar{Y}}{\rho\bar{X} + \beta_2}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{12}^2$	$\frac{(1-f)}{n}\left(R_{12}^2S_x^2 + S_y^2(1-\rho^2)\right)$

It is to be noted that "the existing modified ratio estimators" means the list of modified ratio estimators to be considered in this paper unless otherwise stated. It does not mean to the entire list of modified ratio estimators available in the literature.

The list of modified linear regression type ratio estimators given in Table 1 uses the known values of the parameters like $\overline{X}, C_x, \beta_1, \beta_2, \rho$ and their linear combinations to improve the ratio estimator in estimation of population mean. In this paper an attempt is made to use the Kurtosis, Skewness, Population Correlation coefficient and Quartiles of the auxiliary variable to introduce modified linear regression type ratio estimators for estimating population mean in line with Kadilar and Cingi [4].

2. PROPOSED MODIFIED LINEAR REGRESSION TYPE RATIO ESTIMATORS

In this section, a class of modified linear regression type ratio estimators is proposed for estimating the finite population mean and also derived the bias and the mean squared error of the proposed estimators (see Appendix A). The proposed estimators is defined as $\hat{\overline{Y}}_{p_j}; j = 1,2,\dots,14$ for estimating the population mean $\overline{Y}$ together with the constant, the bias and the mean squared error are presented in the following Table 2:

Table-2. Proposed modified linear regression type ratio estimators with the constant, bias and mean squared error

Estimators	Constant R_{p_j}	Bias-B(.)	MSE (.)
$\hat{\overline{Y}}_{p_1} = [\bar{y} + b(\overline{X} - \bar{x})]\left[\frac{\beta_2\overline{X} + Q_1}{\beta_2\bar{x} + Q_1}\right]$	$R_{p_1} = \frac{\beta_2\overline{Y}}{\beta_2\overline{X} + Q_1}$	$\frac{(1-f)}{n}\frac{S_x^2}{\overline{Y}}R_{p_1}^2$	$\frac{(1-f)}{n}\left(R_{p_1}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\overline{Y}}_{p_2} = [\bar{y} + b(\overline{X} - \bar{x})]\left[\frac{\beta_2\overline{X} + Q_3}{\beta_2\bar{x} + Q_3}\right]$	$R_{p_2} = \frac{\beta_2\overline{Y}}{\beta_2\overline{X} + Q_3}$	$\frac{(1-f)}{n}\frac{S_x^2}{\overline{Y}}R_{p_2}^2$	$\frac{(1-f)}{n}\left(R_{p_2}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\overline{Y}}_{p_3} = [\bar{y} + b(\overline{X} - \bar{x})]\left[\frac{\beta_2\overline{X} + Q_r}{\beta_2\bar{x} + Q_r}\right]$	$R_{p_3} = \frac{\beta_2\overline{Y}}{\beta_2\overline{X} + Q_r}$	$\frac{(1-f)}{n}\frac{S_x^2}{\overline{Y}}R_{p_3}^2$	$\frac{(1-f)}{n}\left(R_{p_3}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\overline{Y}}_{p_4} = [\bar{y} + b(\overline{X} - \bar{x})]\left[\frac{\beta_2\overline{X} + Q_d}{\beta_2\bar{x} + Q_d}\right]$	$R_{p_4} = \frac{\beta_2\overline{Y}}{\beta_2\overline{X} + Q_d}$	$\frac{(1-f)}{n}\frac{S_x^2}{\overline{Y}}R_{p_4}^2$	$\frac{(1-f)}{n}\left(R_{p_4}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\overline{Y}}_{p_5} = [\bar{y} + b(\overline{X} - \bar{x})]\left[\frac{\beta_2\overline{X} + Q_a}{\beta_2\bar{x} + Q_a}\right]$	$R_{p_5} = \frac{\beta_2\overline{Y}}{\beta_2\overline{X} + Q_a}$	$\frac{(1-f)}{n}\frac{S_x^2}{\overline{Y}}R_{p_5}^2$	$\frac{(1-f)}{n}\left(R_{p_5}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\overline{Y}}_{p_6} = [\bar{y} + b(\overline{X} - \bar{x})]\left[\frac{\beta_1\overline{X} + Q_1}{\beta_1\bar{x} + Q_1}\right]$	$R_{p_6} = \frac{\beta_1\overline{Y}}{\beta_1\overline{X} + Q_1}$	$\frac{(1-f)}{n}\frac{S_x^2}{\overline{Y}}R_{p_6}^2$	$\frac{(1-f)}{n}\left(R_{p_6}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\overline{Y}}_{p_7} = [\bar{y} + b(\overline{X} - \bar{x})]\left[\frac{\beta_1\overline{X} + Q_3}{\beta_1\bar{x} + Q_3}\right]$	$R_{p_7} = \frac{\beta_1\overline{Y}}{\beta_1\overline{X} + Q_3}$	$\frac{(1-f)}{n}\frac{S_x^2}{\overline{Y}}R_{p_7}^2$	$\frac{(1-f)}{n}\left(R_{p_7}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$

$\hat{\hat{Y}}_{p_8} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\beta_1\bar{X} + Q_r}{\beta_1\bar{x} + Q_r}\right]$	$R_{p_8} = \frac{\beta_1\bar{Y}}{\beta_1\bar{X} + Q_r}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p_8}^2$	$\frac{(1-f)}{n}\left(R_{p_8}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\hat{Y}}_{p_9} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\beta_1\bar{X} + Q_a}{\beta_1\bar{x} + Q_a}\right]$	$R_{p_9} = \frac{\beta_1\bar{Y}}{\beta_1\bar{X} + Q_a}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p_9}^2$	$\frac{(1-f)}{n}\left(R_{p_9}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\hat{Y}}_{p_{10}} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\rho\bar{X} + Q_1}{\rho\bar{x} + Q_1}\right]$	$R_{p_{10}} = \frac{\rho\bar{Y}}{\rho\bar{X} + Q_1}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p_{10}}^2$	$\frac{(1-f)}{n}\left(R_{p_{10}}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\hat{Y}}_{p_{11}} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\rho\bar{X} + Q_3}{\rho\bar{x} + Q_3}\right]$	$R_{p_{11}} = \frac{\rho\bar{Y}}{\rho\bar{X} + Q_3}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p_{11}}^2$	$\frac{(1-f)}{n}\left(R_{p_{11}}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\hat{Y}}_{p_{12}} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\rho\bar{X} + Q_r}{\rho\bar{x} + Q_r}\right]$	$R_{p_{12}} = \frac{\rho\bar{Y}}{\rho\bar{X} + Q_r}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p_{12}}^2$	$\frac{(1-f)}{n}\left(R_{p_{12}}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\hat{Y}}_{p_{13}} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\rho\bar{X} + Q_d}{\rho\bar{x} + Q_d}\right]$	$R_{p_{13}} = \frac{\rho\bar{Y}}{\rho\bar{X} + Q_d}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p_{13}}^2$	$\frac{(1-f)}{n}\left(R_{p_{13}}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$
$\hat{\hat{Y}}_{p_{14}} = [\bar{y} + b(\bar{X} - \bar{x})]\left[\frac{\rho\bar{X} + Q_a}{\rho\bar{x} + Q_a}\right]$	$R_{p_{14}} = \frac{\rho\bar{Y}}{\rho\bar{X} + Q_a}$	$\frac{(1-f)}{n}\frac{S_x^2}{\bar{Y}}R_{p_{14}}^2$	$\frac{(1-f)}{n}\left(R_{p_{14}}^2 S_x^2 + S_y^2(1 - \rho^2)\right)$

3. EFFICIENCY OF THE PROPOSED ESTIMATORS

The Variance of SRSWOR sample mean $\bar{y}_{srs}$is given below:

$$V(\bar{y}_{srs}) = \frac{(1-f)}{n}S_y^2 \quad (8)$$

The mean squared error of the usual ratio estimator $\hat{\hat{Y}}_R$ to the first degree of approximation is given below:

$$MSE\left(\hat{\hat{Y}}_R\right) = \frac{(1-f)}{n}\bar{Y}^2\left(C_y^2 + C_x^2 - 2\rho C_x C_y\right) \quad (9)$$

The modified ratio estimators given in Table 1 and the proposed modified ratio estimators given in Table 2 are represented in three classes as given below:

Class 1: The mean squared error and the constant of the existing modified linear regression type ratio estimators $\hat{\hat{Y}}_1$ to $\hat{\hat{Y}}_{12}$ listed in the Table 1 are represented in a single class (say, Class 1), which will be very much useful for comparing with that of proposed modified linear regression type ratio estimators and are given below:

$$MSE\left(\hat{\hat{Y}}_i\right) = \frac{(1-f)}{n}\left(R_i^2 S_x^2 + S_y^2(1 - \rho^2)\right); i = 1, 2, 3, \ldots, 12 \quad (10)$$

Class 2:The mean squared error and the constant of the proposed modified linear regression type ratio Estimators $\hat{\hat{Y}}_{p_1}$ to $\hat{\hat{Y}}_{p_{14}}$ listed in the Table 2 are represented in a single class (say, Class

2), which will be very much useful for comparing with that of existing modified linear regression type ratio estimators (given in Class1) are given below:

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(R_{p_j}^2 S_x^2 + S_y^2(1-\rho^2)\right); j = 1,2,3,\ldots,14 \quad (11)$$

From the expressions given in (8) and (11) we have derived the conditions (see Appendix-B) for which the proposed estimator $\hat{\hat{Y}}_{p_j}$ are more efficient than the simple random sampling without replacement (SRSWOR) sample mean $\bar{y}_{srs}$ and are given below:

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) < V(\bar{y}_{srs}) \text{ if } R_{p_j} \leq \rho\frac{S_y}{S_x} \; ; \; j = 1,2,3,\ldots,14 \quad (12)$$

From the expressions given in (9) and (11) we have derived the conditions (see Appendix-C) for which the proposed estimators $\hat{\hat{Y}}_{p_j}$ are more efficient than the usual ratio estimator and are given below:

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) \leq MSE\left(\hat{\hat{Y}}_R\right) \text{ if } \bar{Y}\left(\frac{C_x - \rho C_y}{S_x}\right) \leq R_{p_j} \leq \bar{Y}\left(\frac{\rho C_y - C_x}{S_x}\right)$$

or

$$\bar{Y}\left(\frac{\rho C_y - C_x}{S_x}\right) \leq R_{p_j} \leq \bar{Y}\left(\frac{C_x - \rho C_y}{S_x}\right) \quad (13)$$

From the expressions given in (10) and (11) we have derived the conditions (see Appendix-D) for which the proposed estimators $\hat{\hat{Y}}_{p_j}; j = 1,2,3,\ldots 14$ are more efficient than the existing modified ratio estimators given in Class 1, $\hat{\hat{Y}}_i; i = 1,2,3,\ldots,12$ and are given below:

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) < MSE\left(\hat{\hat{Y}}_i\right) \text{ if } R_{p_j} < R_i \, ; i = 1,2,3,\ldots,12 : j = 1,2,3,\ldots,14 \quad (14)$$

4. EMPIRICAL STUDY

The performance of the proposed modified linear regression type ratio estimators are assessed with that of the SRSWOR sample mean, the usual ratio estimator and the existing modified ratio estimators for a natural population considered by Kadilar and Cingi [4]. The population consists of data on apple production amount (as study variable) and number of apple trees (as auxiliary variable) in 106 villages of Aegean Region in 1999. The parameters computed from the above population are given below:

$N = 106$	$n = 40$	$\bar{Y} = 2212.5943$	$\bar{X} = 27421.6981$
$S_y = 11496.9102$	$C_y = 5.1961$	$S_x = 57188.9320$	$C_x = 2.0855$
$\rho = 0.8560$	$\beta_{2(x)} = 34.5723$	$\beta_{1(x)} = 5.1238$	$b = 0.1721$
$Q_1 = 2387.5$	$Q_3 = 26700$	$Q_r = 24312.5$	$Q_d = 12156.25$
$Q_a = 14543.75$			

The constant, the bias and the mean squared error of the existing and proposed modified ratio estimators are given in Table 3:

Table-3. Constant, Bias and Mean squared error of the existing and proposed estimators

Estimators		Constant	B(.)	MSC (.)
SRSWOR Sample mean	$\bar{y}_{srs}$	-	-	2057484.9094
Ratio Estimator	$\hat{\bar{Y}}_R$	0.0807	169.6823	975171.1057
Linear Regression Estimator	$\hat{\bar{Y}}_{lr}$	-	-	549896.3301
Existing Modified linear regression type ratio estimators (Class 1)	$\hat{\bar{Y}}_1$	0.0807	149.8012	881345.5029
	$\hat{\bar{Y}}_2$	0.0807	149.7784	881295.0933
	$\hat{\bar{Y}}_3$	0.0806	149.4241	880511.3222
	$\hat{\bar{Y}}_4$	0.0807	149.8005	881344.0447
	$\hat{\bar{Y}}_5$	0.0806	149.6202	880945.1187
	$\hat{\bar{Y}}_6$	0.0807	149.7918	881324.8108
	$\hat{\bar{Y}}_7$	0.0807	149.7967	881335.5808
	$\hat{\bar{Y}}_8$	0.0807	149.7745	881286.6144
	$\hat{\bar{Y}}_9$	0.0807	149.8009	881344.9044
	$\hat{\bar{Y}}_{10}$	0.0806	149.3609	880371.3022
	$\hat{\bar{Y}}_{11}$	0.0807	149.7452	881221.6738
	$\hat{\bar{Y}}_{12}$	0.0807	149.7275	881182.4501
Proposed Modified linear regression type ratio estimators (Class 2)	$\hat{\bar{Y}}_{p_1}$	0.0805	149.0495	879,682.3602
	$\hat{\bar{Y}}_{p_2}$	0.0785	141.7068	863,435.9767
	$\hat{\bar{Y}}_{p_3}$	0.0787	142.4035	864,977.6022
	$\hat{\bar{Y}}_{p_4}$	0.0797	146.0321	873,006.1441
	$\hat{\bar{Y}}_{p_5}$	0.0795	145.3086	871,405.2963
	$\hat{\bar{Y}}_{p_6}$	0.0793	144.8370	870,361.9563
	$\hat{\bar{Y}}_{p_7}$	0.0678	105.7788	783,941.8306
	$\hat{\bar{Y}}_{p_8}$	0.0688	108.8656	790,771.6476
	$\hat{\bar{Y}}_{p_9}$	0.0731	123.0159	822,080.5680
	$\hat{\bar{Y}}_{p_{10}}$	0.0732	123.4180	822,970.2420
	$\hat{\bar{Y}}_{p_{11}}$	0.0377	32.7878	622,442.3311
	$\hat{\bar{Y}}_{p_{12}}$	0.0396	36.1459	629,872.6416
	$\hat{\bar{Y}}_{p_{13}}$	0.0532	65.0188	693,756.4581
	$\hat{\bar{Y}}_{p_{14}}$	0.0498	57.1087	676,254.6344

From the values of Table 4, it is observed that mean squared error of the proposed modified linear regression type ratio estimators are less than the variance of SRSWOR sample mean, mean squared error of the usual ratio estimator and existing modified linear regression type ratio estimators. The percent relative efficiencies (PRE's) of the proposed estimators with respect to the existing estimators computed by the formula as given below:

PRE $\left(\widehat{\bar{Y}}_{p_j}\right) = \frac{MSE(.)}{MSE\left(\widehat{\bar{Y}}_{p_j}\right)} * 100; j = 1,2,3, \ldots, 14$ and are presented in the following tables:

Table-4. PREs of the proposed estimators $\widehat{\bar{Y}}_{p_j};\ j = 1,2,3, \ldots, 7$

Estimators	$\widehat{\bar{Y}}_{p_1}$	$\widehat{\bar{Y}}_{p_2}$	$\widehat{\bar{Y}}_{p_3}$	$\widehat{\bar{Y}}_{p_4}$	$\widehat{\bar{Y}}_{p_5}$	$\widehat{\bar{Y}}_{p_6}$	$\widehat{\bar{Y}}_{p_7}$
$\bar{y}_{srs}$	233.89	238.29	237.87	235.68	236.11	236.39	262.45
$\widehat{\bar{Y}}_R$	110.85	112.94	112.74	111.70	111.91	112.04	124.39
$\widehat{\bar{Y}}_1$	100.19	102.07	101.89	100.96	101.14	101.26	112.42
$\widehat{\bar{Y}}_2$	100.18	102.07	101.89	100.95	101.13	101.26	112.42
$\widehat{\bar{Y}}_3$	100.09	101.98	101.80	100.86	101.04	101.17	112.32
$\widehat{\bar{Y}}_4$	100.19	102.07	101.89	100.96	101.14	101.26	112.42
$\widehat{\bar{Y}}_5$	100.14	102.03	101.85	100.91	101.09	101.22	112.37
$\widehat{\bar{Y}}_6$	100.19	102.07	101.89	100.95	101.14	101.26	112.42
$\widehat{\bar{Y}}_7$	100.19	102.07	101.89	100.95	101.14	101.26	112.42
$\widehat{\bar{Y}}_8$	100.18	102.07	101.89	100.95	101.13	101.26	112.42
$\widehat{\bar{Y}}_9$	100.19	102.07	101.89	100.96	101.14	101.26	112.42
$\widehat{\bar{Y}}_{10}$	100.08	101.96	101.78	100.84	101.03	101.15	112.30
$\widehat{\bar{Y}}_{11}$	100.17	102.06	101.88	100.94	101.13	101.25	112.41
$\widehat{\bar{Y}}_{12}$	100.17	102.06	101.87	100.94	101.12	101.24	112.40

Table-5.PREs of the proposed estimators $\widehat{\bar{Y}}_{p_j};\ j = 8,9,10, \ldots, 14$

Estimators	$\widehat{\bar{Y}}_{p_8}$	$\widehat{\bar{Y}}_{p_9}$	$\widehat{\bar{Y}}_{p_{10}}$	$\widehat{\bar{Y}}_{p_{11}}$	$\widehat{\bar{Y}}_{p_{12}}$	$\widehat{\bar{Y}}_{p_{13}}$	$\widehat{\bar{Y}}_{p_{14}}$
$\bar{y}_{srs}$	260.19	250.28	250.01	330.55	326.65	296.57	304.25
$\widehat{\bar{Y}}_R$	123.32	118.62	118.49	156.67	154.82	140.56	144.20
$\widehat{\bar{Y}}_1$	111.45	107.21	107.09	141.59	139.92	127.04	130.33
$\widehat{\bar{Y}}_2$	111.45	107.20	107.09	141.59	139.92	127.03	130.32
$\widehat{\bar{Y}}_3$	111.35	107.11	106.99	141.46	139.79	126.92	130.20
$\widehat{\bar{Y}}_4$	111.45	107.21	107.09	141.59	139.92	127.04	130.33
$\widehat{\bar{Y}}_5$	111.40	107.16	107.04	141.53	139.86	126.98	130.27
$\widehat{\bar{Y}}_6$	111.45	107.21	107.09	141.59	139.92	127.04	130.32
$\widehat{\bar{Y}}_7$	111.45	107.21	107.09	141.59	139.92	127.04	130.33
$\widehat{\bar{Y}}_8$	111.45	107.20	107.09	141.59	139.92	127.03	130.32
$\widehat{\bar{Y}}_9$	111.45	107.21	107.09	141.59	139.92	127.04	130.33
$\widehat{\bar{Y}}_{10}$	111.33	107.09	106.97	141.44	139.77	126.90	130.18
$\widehat{\bar{Y}}_{11}$	111.44	107.19	107.08	141.57	139.90	127.02	130.31
$\widehat{\bar{Y}}_{12}$	111.43	107.19	107.07	141.57	139.90	127.02	130.30

From the values of Table 4 and Table 5, it is observed that the PRE's of the proposed estimators with respect to the existing estimators are ranging from 100.08 to 330.55. This shows that proposed estimators are more efficient and perform better than the SRSWOR sample mean, the usual ratio estimator and the existing modified linear regression type ratio estimators.

5. CONCLUSION

In this paper we have proposed a class of modified ratio estimators using the Kurtosis, Skewness, Correlation coefficient and quartiles of the auxiliary variable. The bias and mean squared error of the proposed estimators are obtained and compared with that of the existing estimators. Further we have derived the conditions for which the proposed estimators are more efficient than the SRSWOR sample mean, the usual ratio estimator and the existing modified linear regression type ratio estimators. We have also assessed the performance of the proposed estimators with that of the existing estimators for a natural population. From the numerical study it is observed that the mean squared error of the proposed estimators are less than variance of the SRSWOR sample mean and the mean squared error of the usual ratio estimator and the existing modified linear regression type ratio estimators. Hence we strongly recommend that the proposed modified linear regression type ratio estimators may be preferred over the existing estimators for the use of practical applications.

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Appendix-A

We have derived the bias and mean squared error of the proposed modified linear regression type ratio estimator $\hat{\hat{\bar{Y}}}_{p_j}; j = 1,2,3,\ldots,14$ to first order of approximation and are given below:

Let $e_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$ and $e_1 = \frac{\bar{x}-\bar{X}}{\bar{X}}$. Further we can write $\bar{y} = \bar{Y}(1+e_0)$ and $\bar{x} = \bar{X}(1+e_1)$ and from the definition of e_0 and e_1 we obtain:

$E[e_0] = E[e_1] = 0$

$E[e_0^2] = \frac{(1-f)}{n}C_y^2;\ E[e_1^2] = \frac{(1-f)}{n}C_x^2;\ E[e_0e_1] = \frac{(1-f)}{n}\rho C_y C_x$

The proposed estimators $\hat{\hat{\bar{Y}}}_{p_j}$ in the form of e_0 and e_1 is given below:

$$\hat{\hat{\bar{Y}}}_{p_j} = \bar{Y}(1+e_0) + b(\bar{X}-\bar{x})\left[\frac{(\bar{X}+\lambda_j)}{(\bar{X}(1+e_1)+\lambda_j)}\right]; j = 1,2,3,\ldots,14$$

where $\lambda_1 = \frac{Q_1}{\beta_2}, \lambda_2 = \frac{Q_3}{\beta_2}, \lambda_3 = \frac{Q_r}{\beta_2}, \lambda_4 = \frac{Q_d}{\beta_2}, \lambda_5 = \frac{Q_a}{\beta_2}, \lambda_6 = \frac{Q_1}{\beta_1}, \lambda_7 = \frac{Q_3}{\beta_1}, \lambda_8 = \frac{Q_r}{\beta_1}$

$$\lambda_9 = \frac{Q_a}{\beta_1}, \lambda_{10} = \frac{Q_1}{\rho}, \lambda_{11} = \frac{Q_3}{\rho}, \lambda_{12} = \frac{Q_r}{\rho}, \lambda_{13} = \frac{Q_d}{\rho}, \lambda_{14} = \frac{Q_a}{\rho}$$

$$\hat{\hat{\bar{Y}}}_{p_j} = \bar{Y}(1+e_0) + b(\bar{X}-\bar{x})\left[\frac{(\bar{X}+\lambda_j)}{(\bar{X}+\lambda_j)\left(1+\frac{e_1\bar{X}}{\bar{X}+\lambda_j}\right)}\right]$$

$$\Rightarrow \hat{\hat{\bar{Y}}}_{p_j} = \frac{\bar{Y}(1+e_0) + b(\bar{X}-\bar{x})}{\left(1+\theta_{p_j}e_1\right)} \text{ where } \theta_{p_j} = \frac{\bar{X}}{\bar{X}+\lambda_j}; j = 1,2,3,\ldots,14$$

$$\Rightarrow \hat{\hat{\bar{Y}}}_{p_j} = \bar{Y}(1+e_0) + b(\bar{X}-\bar{x})\left(1+\theta_{p_j}e_1\right)^{-1}$$

$$\Rightarrow \hat{\hat{\bar{Y}}}_{p_j} = \bar{Y}(1+e_0) + b(\bar{X}-\bar{x})\left(1-\theta_{p_j}e_1+\theta_{p_j}^2e_1^2-\theta_{p_j}^3e_1^3+\cdots\right)$$

Neglecting the terms higher than third order, we will get:

$$\hat{\hat{\bar{Y}}}_{p_j} = \bar{Y} + \bar{Y}e_0 - b\bar{X}e_1 - \bar{Y}\theta_{p_j}e_1 - \bar{Y}\theta_{p_j}e_0e_1 + b\bar{X}\theta_{p_j}e_1^2 + \bar{Y}\theta_{p_j}^2e_1^2$$

$$\Rightarrow \hat{\hat{\bar{Y}}}_{p_j} - \bar{Y} = \bar{Y}e_0 - b\bar{X}e_1 - \bar{Y}\theta_{p_j}e_1 - \bar{Y}\theta_{p_j}e_0e_1 + b\bar{X}\theta_{p_j}e_1^2 + \bar{Y}\theta_{p_j}^2e_1^2 \qquad \text{(A)}$$

Taking expectation on both sides of (A), we get:

$$E\left(\hat{\bar{Y}}_{p_j}-\bar{Y}\right)=\bar{Y}E(e_0)+b\bar{X}E(e_1)-\bar{Y}\theta_{p_j}E(e_1)-\bar{Y}\theta_{p_j}E(e_0e_1)+b\bar{X}\theta_{p_j}E(e_1^2)+\bar{Y}\theta_{p_j}^2E(e_1^2)$$

$$\text{Bias}\left(\hat{\bar{Y}}_{p_j}\right)=\frac{(1-f)}{n}\left(\bar{Y}\theta_{p_j}^2C_x^2-\theta_{p_j}\frac{S_{xy}}{S_x}C_x+\theta_{p_j}\frac{S_{xy}}{S_x}C_x\right)$$

$$\text{Bias}\left(\hat{\bar{Y}}_{p_j}\right)=\frac{(1-f)}{n}\left(\bar{Y}\theta_{p_j}^2C_x^2\right);\, j=1,2,3,\dots,14 \qquad \text{(B)}$$

Substitute the value of θ_{p_j} in (B), we get the bias of the proposed estimator $\hat{\bar{Y}}_{p_j}; j=1,2,3,\dots,10$ as given below:

$$\text{Bias}\left(\hat{\bar{Y}}_{p_j}\right)=\frac{(1-f)}{n}\left(\bar{Y}\frac{\bar{X}^2}{(\bar{X}+\lambda_j)^2}C_x^2\right) \qquad \text{(C)}$$

Multiply and divide by $\bar{Y}^2$ in (3.19), we get

$$\text{Bias}\left(\hat{\bar{Y}}_{p_j}\right)=\frac{(1-f)}{n}\left(\frac{S_x^2}{\bar{Y}}R_{p_j}^2\right) \text{ where } R_{p_j}=\frac{\bar{Y}}{\bar{X}+\lambda_j};\ j=1,2,3,\dots,14 \qquad \text{(D)}$$

Squaring both sides of (A), neglecting the terms more than 2nd order and taking expectation on both sides we get

$$E\left(\hat{\bar{Y}}_{p_j}-\bar{Y}\right)^2$$
$$=E(\bar{Y}^2e_0^2)+b^2\bar{X}^2E(e_1^2)+\bar{Y}^2\theta_{p_j}^2E(e_1^2)-2b\bar{Y}\bar{X}E(e_0e_1)$$
$$-2\bar{Y}^2\theta_{p_j}E(e_0e_1)+2b\bar{X}\bar{Y}\theta_{p_j}E(e_1^2)$$

$$\text{MSE}\left(\hat{\bar{Y}}_{p_j}\right)$$
$$=\frac{(1-f)}{n}\Big(\bar{Y}^2C_y^2+b^2\bar{X}^2C_x^2+\bar{Y}^2\theta_{p_j}^2C_x^2-2b\bar{Y}\bar{X}\rho C_yC_x$$
$$-2\bar{Y}^2\theta_{p_j}\rho C_yC_x+2b\bar{X}\bar{Y}\theta_jC_x^2\Big)$$

$$\text{MSE}\left(\hat{\bar{Y}}_{p_j}\right)$$
$$=\frac{(1-f)}{n}\Bigg(\bar{Y}^2C_y^2+\frac{{S_{xy}}^2}{(S_x^2)^2}S_x^2+\bar{Y}^2\theta_{p_j}^2C_x^2-2\frac{S_{xy}}{S_x^2}\frac{S_{xy}}{S_xS_y}S_yS_x$$
$$-2\bar{Y}\theta_{p_j}\frac{S_{xy}}{S_xS_y}S_yC_x+2\frac{S_{xy}}{S_x^2}S_x\bar{Y}\theta_{p_j}C_x\Bigg)$$

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(\bar{Y}^2C_y^2 + \bar{Y}^2\theta_{p_j}^2C_x^2 + \frac{S_{xy}^2}{S_x^2} - 2\frac{S_{xy}^2}{S_x^2} - 2\bar{Y}\theta_j\frac{S_{xy}}{S_x}C_x + 2\bar{Y}\theta_{p_j}\frac{S_{xy}}{S_x}C_x\right)$$

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(\bar{Y}^2\theta_{p_j}^2C_x^2 + \bar{Y}^2C_y^2 - \frac{S_{xy}^2}{S_x^2}\right)$$

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(\bar{Y}^2\theta_{p_j}^2C_x^2 + S_y^2 - \frac{\rho^2S_x^2S_y^2}{S_x^2}\right) \text{ since } \rho = \frac{S_{xy}}{S_xS_y}$$

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(\bar{Y}^2\theta_{p_j}^2C_x^2 + S_y^2 - \rho^2S_y^2\right)$$

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(\bar{Y}^2\theta_{p_j}^2C_x^2 + S_y^2(1-\rho^2)\right)$$

Substitute the value of θ_{p_j} in the above expression, we will get:

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(\bar{Y}^2\frac{\bar{X}^2}{\left(\bar{X}+\lambda_j\right)^2}C_x^2 + S_y^2(1-\rho^2)\right)$$

$$\Rightarrow MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(\frac{\bar{Y}^2}{\left(\bar{X}+\lambda_j\right)^2}\bar{X}^2C_x^2 + S_y^2(1-\rho^2)\right)$$

The mean squared error of the proposed modified linear regression type ratio estimators $\hat{\hat{Y}}_{p_j}$ is given below:

$$MSE\left(\hat{\hat{Y}}_{p_j}\right) = \frac{(1-f)}{n}\left(R_{p_j}^2S_x^2 + S_y^2(1-\rho^2)\right) \text{ where } R_{p_j} = \frac{\bar{Y}}{\bar{X}+\lambda_j}; j = 1,2,3,\ldots,14 \qquad \text{(E)}$$

Appendix-B

The conditions for which the proposed modified linear regression type ratio estimators (Class 2) perform better than the SRSWOR sample mean are derived and are given below:

For $MSE\left(\hat{\hat{Y}}_{p_j}\right) \leq V(\bar{y}_{srs})$

$$\frac{(1-f)}{n}\left(R_{p_j}^2S_x^2 + S_y^2(1-\rho^2)\right) \leq \frac{(1-f)}{n}S_y^2$$

$$\Rightarrow \left(R_{p_j}^2S_x^2 + S_y^2(1-\rho^2)\right) \leq S_y^2$$

$$\Rightarrow R_{p_j}^2S_x^2 + S_y^2 - S_y^2\rho^2 - S_y^2 \leq 0$$

$$\Rightarrow R_{pj}^2 S_x^2 - S_y^2 \rho^2 \leq 0$$
$$\Rightarrow R_{pj}^2 S_x^2 \leq S_y^2 \rho^2$$
$$\Rightarrow R_{pj} S_x \leq S_y \rho$$
$$\Rightarrow \rho \geq R_{pj} \frac{S_x}{S_y}$$
$$\Rightarrow R_{pj} \leq \rho \frac{S_y}{S_x}$$

That is $MSE\left(\hat{\hat{\bar{Y}}}_{pj}\right) \leq V(\bar{y}_{srs})$ if $R_{pj} \leq \rho \frac{S_y}{S_x}$

Appendix-C

The conditions for which the proposed modified ratio estimators (Class 2) perform better than the usual ratio estimator are derived and are given below:

For $MSE\left(\hat{\hat{\bar{Y}}}_{pj}\right) \leq MSE\left(\hat{\bar{Y}}_R\right)$

$$\frac{(1-f)}{n}\left(R_{pj}^2 S_x^2 + S_y^2(1-\rho^2)\right) \leq \frac{(1-f)}{n}\bar{Y}^2\left(C_y^2 + C_x^2 - 2\rho C_x C_y\right)$$

$$\Rightarrow \bar{Y}^2\left(R_{pj}^{*2} S_x^2 + C_y^2(1-\rho^2)\right) \leq \bar{Y}^2\left(C_y^2 + C_x^2 - 2\rho C_x C_y\right) \text{where } R_{pj}^* = \frac{R_{pj}}{\bar{Y}}$$

$$\Rightarrow R_{pj}^{*2} S_x^2 + C_y^2(1-\rho^2) - C_y^2 - C_x^2 + 2\rho C_x C_y \leq 0$$

$$\Rightarrow R_{pj}^{*2} S_x^2 + C_y^2 - \rho^2 C_y^2 - C_y^2 - C_x^2 + 2\rho C_x C_y \leq 0$$

$$\Rightarrow R_{pj}^{*2} S_x^2 - \rho^2 C_y^2 - C_x^2 + 2\rho C_x C_y \leq 0$$

$$\Rightarrow \left(C_x - \rho C_y\right)^2 - R_{pj}^{*2} S_x^2 \geq 0$$

$$\Rightarrow \left(C_x - \rho C_y + R_{pj}^* S_x\right)\left(C_x - \rho C_y - R_{pj}^* S_x\right) \geq 0$$

Condition 1: $\left(C_x - \rho C_y + R_{pj}^* S_x\right) \leq 0$ and $\left(C_x - \rho C_y - R_{pj}^* S_x\right) \leq 0$

$$C_x + R_{pj}^* S_x \leq \rho C_y \text{ and } C_x - R_{pj}^* S_x \leq \rho C_y$$

$$\Rightarrow R_{pj}^* \leq \frac{\rho C_y - C_x}{S_x} \text{ and } R_{pj}^* \geq \frac{C_x - \rho C_y}{S_x}$$

$$\Rightarrow \frac{C_x - \rho C_y}{S_x} \leq R_{pj}^* \leq \frac{\rho C_y - C_x}{S_x} \text{ where } R_{pj}^* = \frac{R_{pj}}{\bar{Y}}$$

$$\Rightarrow \bar{Y}\left(\frac{C_x - \rho C_y}{S_x}\right) \leq R_{pj} \leq \bar{Y}\left(\frac{\rho C_y - C_x}{S_x}\right)$$

Condition 2: $\left(C_x - \rho C_y + R^*_{pj} S_x\right) \geq 0$ and $\left(C_x - \rho C_y - R^*_{pj} S_x\right) \geq 0$

$$C_x + R^*_{pj} S_x \geq \rho C_y \text{ and } C_x - R^*_{pj} S_x \geq \rho C_y$$

$$\Rightarrow R^*_{pj} \geq \frac{\rho C_y - C_x}{S_x} \text{ and } R^*_{pj} \leq \frac{C_x - \rho C_y}{S_x}$$

$$\Rightarrow \frac{\rho C_y - C_x}{S_x} \leq R^*_{pj} \leq \frac{C_x - \rho C_y}{S_x}$$

$$\Rightarrow \bar{Y}\left(\frac{\rho C_y - C_x}{S_x}\right) \leq R_{pj} \leq \bar{Y}\left(\frac{C_x - \rho C_y}{S_x}\right)$$

That is $MSE\left(\hat{\hat{Y}}_{pj}\right) \leq MSE\left(\hat{\hat{Y}}_R\right)$ if $\bar{Y}\left(\frac{C_x - \rho C_y}{S_x}\right) \leq R_{pj} \leq \bar{Y}\left(\frac{\rho C_y - C_x}{S_x}\right)$

or

$$\bar{Y}\left(\frac{\rho C_y - C_x}{S_x}\right) \leq R_{pj} \leq \bar{Y}\left(\frac{C_x - \rho C_y}{S_x}\right)$$

Appendix-D

The conditions for which the proposed modified linear regression type ratio estimators (Class 2) perform better than the existing modified linear regression type ratio estimators (Class 1) are derived and are given below:

For $MSE\left(\hat{\hat{Y}}_{pj}\right) \leq MSE\left(\hat{\hat{Y}}_i\right)$; $i = 1,2,3, \ldots 12$ and $j = 1,2,3, \ldots 14$

$$\frac{(1-f)}{n}\left(R^2_{pj} S^2_x + S^2_y (1-\rho^2)\right) \leq \frac{(1-f)}{n}\left(R^2_i S^2_x + S^2_y (1-\rho^2)\right)$$

$$\Rightarrow R^2_{pj} S^2_x + S^2_y (1-\rho^2) \leq R^2_i S^2_x + S^2_y (1-\rho^2)$$

$$\Rightarrow R^2_{pj} S^2_x \leq R^2_i S^2_x$$

$$\Rightarrow R^2_{pj} \leq R^2_i$$

$$\Rightarrow R_{pj} \leq R_i$$

That is $MSE\left(\hat{\hat{Y}}_{pj}\right) < MSE\left(\hat{\hat{Y}}_i\right)$ if $R_{pj} < R_i$; $i = 1, 2, 3, \ldots, 12$: $j = 1, 2, 3, \ldots, 14$