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A GENERAL MEASURE FOR THE RELATIVE EFFICIENCY OF ANY TWO SCORING SYSTEMS

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ABSTRACT

Miles (1984) developed a very elegant theory for the relative efficiency of different scoring systems at correctly identifying the better player, assuming points were independent. This earlier work was limited to those situations in which the underlying probability structures of the game being modelled had certain restrictive characteristics. Using those underlying characteristics it was possible to use interpolation methods to derive efficiency measures in a restricted number of practical situations. The major objective of this research was to investigate whether Miles' work on the efficiency of scoring systems could be extended to more general situations. Games that do not possess the restrictive probability structures noted above have been considered, and it has been shown that an extrapolation method for deriving efficiency measures can be developed and applied. In doing so the efficiency of nested scoring systems has been studied. It turns out that this extrapolation method can be used in any scoring system situation, even where the outcome is win/draw/loss rather than win/loss. It produces exactly the same efficiency formula as that produced by the interpolated method. Thus, the method for measuring efficiency has been extended to a wider range of practical scoring systems situations.

Contribution/ Originality

This study shows how the relative efficiency of any two scoring systems can be evaluated, thus considerably extending earlier work in which efficiency could be evaluated only in a limited number of situations.

1. INTRODUCTION

Miles (1984) developed a very elegant theory for the relative efficiency of different scoring systems, assuming that points were independent. He considered 'win-by-n' (Wn) scoring systems in which the winner was the first player to win n more points than his opponent. Assuming points are independent and player A has a constant probability p of winning every point (unipoints), it can be shown that the probability P that player A wins Wn, the expected number of points played μ, and n satisfy the '(P, μ, n) equation'

$$(P-Q)/\mu = (p-q)/n$$

and that the ratio P/Q is given by

$$(P/Q) = (p/q)^n$$

where Q = 1 − P and q = 1 - p. These equations follow from the fact that W_n has the constant probability ratio (cpr) property (Pollard, 1990). That is, the ratio of the probability that player A wins in n + 2m points divided by the probability that he loses in n + 2m points (m = 0, 1, 2, …) is constant, and is equal to P/Q. Noting the optimal nature of this Wn system (Wald and Wolfowitz, 1948), and using Wn (points) as the family of scoring systems with unit efficiency, Miles (1984) showed that the efficiency ρ of a general 'unipoints' scoring system SS with key characteristics P and μ is given by

$$\rho = \frac{(P-Q)\ln(P/Q)}{\mu(p-q)\ln(p/q)} \tag{1}$$

This efficiency measure, as described by Miles (p. 97), is defined as the expected duration of the 'interpolated' Wn system with the same P-value as SS (as derived from the above '(P, μ, n) equation') divided by the expected duration of SS, namely μ. Note that the value of n for this 'interpolated' Wn system is given by

$$n = \ln(P/Q)/\ln(p/q),$$

resulting from the cpr property of Wn.

It follows, by ignoring the factors involving p (and q) in (1) above, that the expression

$$((P-Q)/\mu)\ln(P/Q)$$

is the measure for the relative efficiency *of a unipoints scoring system given underlying independent points.*

Miles (1984) also considered scoring systems relevant to tennis (and other sports such as volleyball), which he called 'bipoints' scoring systems. He assumed that the probability player A (B) wins a point on service is p_a (p_b), and that points are independent. Noting the work of Wald (1947) and using Wn (point-pairs) as the standard family of scoring systems with unit efficiency,

he showed that the efficiency of a general bipoints scoring system with key characteristics P and μ is given by

$$\rho = \frac{2(P-Q)\ln(P/Q)}{\mu(p_a - p_b)\ln(p_a q_b / p_b q_a)} \qquad (2)$$

where $q_a = 1 - p_a$ and $q_b = 1 - p_b$. As in the unipoints Wn case, the Wn(point-pairs) family of scoring systems possesses a '(P, μ, n) equation' and a cpr property, leading directly to equation (2) by using the same 'interpolation' method as that used above for unipoints.

Thus, ignoring the constant and the factors involving p_a and p_b, the measure for relative efficiency is *also* given by

$$((P-Q)/\mu)\ln(P/Q)$$

for this bipoints system *with underlying independent points with constant p-values p_a and p_b*. That is, the efficiency of the bipoints scoring system 1 relative to the bipoints scoring system 2 is given by

$$\frac{((P_1 - Q_1)/\mu_1)\ln(P_1/Q_1)}{((P_2 - Q_2)/\mu_2)\ln(P_2/Q_2)} \qquad (3)$$

using an obvious notation.

Pollard and Pollard (2008) used the above interpolation method to show that (3) is also the measure for relative efficiency for the independent quad-points case (e.g. tennis doubles with parameters p_{a1}, p_{a2}, p_{b1} and p_{b2}). Further, they showed Pollard and Pollard (2010), again using the interpolation approach, that (3) is the relative efficiency expression for scoring systems where unipoints or bipoints become one-step dependent probabilities. It was possible to derive the relative efficiency for each of these four situations (unipoints, bipoints, quadpoints (e.g. tennis doubles), 1-step dependent unipoints and bipoints) because in each case the underlying point probability structure of the situation being modelled lead to both a '(P, μ, n) equation' and cpr property for the relevant underlying Wn system. This however is not always the case, and the interpolation approach is not possible when it is not. For example, supposing player A has a probability p of winning a point when the players are equal, p+ when he is ahead, and p- when he is behind, it can easily be seen that the Wn system of scoring systems does not have the cpr property when n > 2.

In this paper we consider an alternative approach to relative efficiency. This alternative approach does not depend on there being a '(P, μ, n) equation' (and a cpr within an underlying Wn system) that is necessary for the interpolation method.

2. METHODS

Miles (1984) noted that efficiency under nesting was 'roughly multiplicative' (p. 107). An aspect of this approximation is that the expected number of points in a set of tennis is only approximately equal to the expected number of points in a game (different for each player)

multiplied by the expected number of games in a set (Pollard, 1983). Further, the expected number of service games for one player is typically different to the expected number for the other. *We now consider several examples in which efficiency under nesting is exactly multiplicative.*

Example-1

Suppose player A has a constant probability 0.6 of winning a point and that points are independent. Consider the nested system B3 (B3), where the outer nest represents a 'set', and the inner nest represents a 'game'. Note that the nested system can be won or lost by player A in as few as 4 points, or as many as 9 points. First principles can be used to show that this nested system has a mean of 6.09135616 points, a probability player A wins of 0.715516416, giving an efficiency of 0.8048199542 (by using (1)). The inner nest has a mean of 2.48 points, and a probability that player A wins of 0.648, giving an efficiency of 0.8981959879 (using (1)), whilst the outer nest with a p-value of 0.648 has a mean of 2.456192 games, and a probability player A wins of 0.715516416, giving an efficiency of 0.8960404689. It can be seen that the product of these efficiencies for the inner and outer nests is exactly equal to the efficiency of the total system, as calculated above.

The efficiency of the inner nest can be expressed as

$$\rho_i = \frac{(P_i - Q_i)\ln(P_i / Q_i)}{\mu_i(p-q)\ln(p/q)}$$

and the efficiency of the outer nest as

$$\rho_o = \frac{(P_o - Q_o)\ln(P_o / Q_o)}{\mu_o(P_i - Q_i)\ln(P_i / Q_i)}$$

using obvious notations, whilst the efficiency of the total nested scoring system is

$$\rho_n = \frac{(P_n - Q_n)\ln(P_n / Q_n)}{\mu_n(p-q)\ln(p/q)}$$

Noting that Pn is the same as Po (and Qn is the same as Qo), it follows that $\rho_n = \rho_i * \rho_o$ if $\mu_n = \mu_i * \mu_o$.

Thus, *the efficiency of a nested system is exactly multiplicative when the expected duration of the nested system is exactly equal to the product of the expected durations of the nests. It is clear that this also applies to triple nesting, etc.*

Some other unipoints examples where efficiency under nesting is exactly multiplicative are other B2n-1(B2m-1) systems such as B3(B5), Wn(B2n-1) systems such as W2(B3), and Wn(Wm) systems such as W2(W3).

It is clear that 'exact multiplicative efficiency' also applies in bipoints, quadpoints, etcetera, when the 'means are multiplicative', as the form of the efficiency expressions remains the same.

Example-2

We consider the nested system Wn(SS) where SS is a scoring system with probability player A wins equal to p, mean duration equal to μ points, mean duration conditional on player A

winning equal to μ_W points and mean duration conditional on player A losing equal to μ_L. Suppose D_z is the expected number of points remaining in the nested system when z is the score of the outer nest (z = -n, -n + 1, -n + 2,…, n - 1, n) and an inner nest is about to begin. It is clear that $D_n = 0$ and $D_{-n} = 0$ are boundary conditions. We have the recurrence relations

$$D_z = p(D_{z+1} + \mu_W) + q(D_{z-1} + \mu_L) \text{ i.e. } D_z = pD_{z+1} + qD_{z-1} + \mu$$

where q = 1 - p. It follows using the methods described in Feller (1957) that

$$D_z = \frac{\mu(z-n)}{q-p} + \frac{2n\mu((q/p)^n - (q/p)^z)}{(q-p)((q/p)^n - (q/p)^{-n})}$$

Putting z = 0, it follows that the expected duration of the nested system is given by

$$D_0 = \frac{P-Q}{p-q}n\mu$$

where

$$P = p^n /(p^n + q^n)$$

and

$$Q = 1 - P.$$

Thus, *the mean of the nested system Wn(SS) is equal to the mean of the inner nest SS multiplied by the mean of outer nest (i.e. the mean of Wn at p). Also, the efficiency of Wn(SS) is equal to the efficiency of SS, since the efficiency of the Wn system is unity.*

Example 3

As a special case of Example 2, W4(B3) with point probability 0.6 is considered. In this case p = 0.648, q = 0.352, μ = 2.48 and D_0 = 28.14489049 using the above equation. This value of D_0 was verified using standard recurrence methods with inner nest conditional means of μ_W = 2.444444 points and μ_L = 2.545454 points. (In the process of considering the state of the outer nest after every second inner nest was completed, it is noted (as a bi-product) that D_2 was equal to 15.53354219 points and D_{-2} was equal to 30.85308066 points, and these values agree with the above equation for D_z.)

Example 3 is a unipoints one. A bipoints example in which the mean of the nested system is exactly equal to the product of the mean of the inner nest and the mean of the outer nest, would be a 'set' of tennis defined as W_2(TB) where TB is the usual tiebreak game.

The above expression for D_0 is used in the following section to extend our definition of efficiency to the situation in which a '(P, μ, n) equation' does not exist for the underlying probabilistic structure under consideration.

2.1. Extended Definition of Relative Efficiency

Suppose the two scoring systems SS1 and SS2 have identical underlying probabilistic structures, and that SSi has an expected duration of μ_i points and a probability that player A wins of p_i (i = 1, 2).

Consider the nested scoring systems Wn1 (SS1) and Wn2(SS2). The probability player A wins Wni(SSi), Pi (i = 1, 2), can be evaluated using the relationship

$$P_i / Q_i = (p_i / q_i)^{n_i}$$

where $P_i + Q_i = 1$ and $p_i + q_i = 1,$

and the expected duration of Wni(SSi) is equal to

$$((P_i - Q_i)/(p_i - q_i))n_i \mu_i ,$$

using the above equation for D_0.

Now suppose n1 and n2 are two (possibly very large) values such that player A has the same probability of winning under either nested system.
That is,

$$P_1 = P_2 \text{ and hence } P_1 / Q_1 = P_2 / Q_2 \text{, and}$$

$$P_1 - Q_1 = P_2 - Q_2 .$$

It follows that

$$n_1 / n_2 = (\ln(p_2 / q_2))/\ln(p_1 / q_1) .$$

Using the underlying concept of efficiency and noting that $P_1 = P_2$ for the two nested systems, the efficiency of the system Wn1(SS1) relative to the system Wn2(SS2) is given by the mean of Wn2(SS2) divided by the mean of Wn1(SS1). That is, it is given by the expression

$$\frac{((p_1 - q_1)/\mu_1)\ln(p_1 / q_1)}{((p_2 - q_2)/\mu_2)\ln(p_2 / q_2)} . \qquad (4)$$

Since in general the efficiency of Wn(SS) is equal to that of SS, it follows that *the efficiency of the system SS1 relative to SS2 is given by expression (4)*. Thus, the expression for the relative efficiency for this case (where a '(p, μ, n) equation' does not necessarily exist) is identical to (3) (for the case when the '(p, μ, n) equation' and the cpr do exist). That is, our measure of relative efficiency is no longer limited to the situation where the underlying probability point structure necessarily allows a Wn system with the cpr property and a '(p, μ, n) equation' to be established. Thus, *the relative efficiency of two systems can now be measured in a much broader range of situations than earlier (and using the same expression)*. In comparison to the 'interpolation approach' to relative efficiency, *the above approach might be called the 'extrapolation' approach to relative efficiency*.

Example 4

In this example the interpolation and extrapolation methods are shown to give identical results. Suppose player A's probability of winning a point is 0.6 and points are independent. For the scoring system SS1 = best of 3 points = B3, we have $p_1 = 0.648$ and $\mu_1 = 2.48$ points and for the scoring system SS2 = best of 5 points = B5, we have $p_2 = 0.68256$ and $\mu_2 = 4.0656$ points, and the above equation becomes

$$n_1 / n_2 = 1.254485322.$$

Thus, as an example of two nested scoring systems, player A has the same probability of winning W1,000,000,000(B5) as he does of winning W1,254,485,322 (B3). [These particular nested scoring systems with very very large expected durations have been chosen so that we have (more than) ample accuracy to satisfactorily demonstrate the extrapolation approach. Note that in practice we don't need n1 and n2 to be anywhere near as large in order to achieve sufficient accuracy.] Using the above equation for D_0, W1,254,485,322 (B3) has expected duration

$$((P_1 - Q_1)/(0.648 - 0.352)) * 3.111123599 * 10^9$$

points, and W1,000,000,000(B5) has expected duration

$$((P_2 - Q_2)/(0.68256 - 0.31744)) * 4.0656 * 10^9$$

points where

$$P_1 - Q_1 = P_2 - Q_2 \text{ are each of course extremely close to unity.}$$

Thus, the efficiency of W1,000,000,000(B5) relative to W1,254,485,322(B3), and hence the efficiency of B5 relative to B3, when the point probability is 0.6, being the ratio of the above two expected durations, is equal to 0.943922915.

It can be seen that, using the interpolation approach, the relative efficiency expression

$$((P - Q)/\mu)\ln(P/Q)$$

is equal to 0.07283742667 for B3, and is equal to 0.06875291605 for B5, so that the efficiency of B5 relative to B3, given by the ratio of these two numbers, is equal to 0.943922915 when p = 0.6, in agreement with the above calculations. Thus, the extrapolation method and the interpolation method give identical results.

Example 5

In this example the interpolation approach to efficiency is not available, but we can use the 'extrapolation method'.

Suppose player A has a probability 0.7 of winning a point when ahead, a probability 0.6 of winning a point when equal, and a probability 0.5 of winning a point when behind. Given this underlying probability structure, it can be seen that the associated family of scoring systems W3,

W4, …does not have a general '(P, μ, n) equation' nor the cpr property. For example, for W3, the probability player A wins in 3 points divided by the probability he loses in 3 points is equal to 2.94, whereas the probability player A wins in 5 points divided by the probability he loses in 5 points is equal to 2.75333, indicating that a cpr does not exist for this underlying probability structure.

Given the above underlying probability structure when player A is ahead, equal and behind, the probability player A wins a best of 3 points game is equal to 0.648 and the expected duration of the game is 2.38 points. Also, the probability he wins a best of 5 points game is equal to 0.68112 and the expected duration of the game is 3.8382 points. Thus the expression

$$((P-Q)/\mu)\ln(P/Q)$$

is equal to 0.07589782275 for B3, and it is equal to 0.07162537163 for B5, and so the efficiency of B5 relative to B3 is equal to 0.9437078566. Note that this relative efficiency is a slightly different value to that in the previous example, not surprisingly as the underlying probability values and structures are slightly different.

2.2. Further Extension of Relative Efficiency to General Win-Draw-Loss Scoring Systems

We now consider two scoring systems SS1 and SS2 which have expected durations μ_1 and μ_2 points and which can result in a win, a draw, or a loss to player A with probabilities p_i, d_i, q_i respectively ($p_i + d_i + q_i = 1$, $i = 1, 2$). Each of these two scoring systems can be converted to one which must result in a win or a loss to player A by repeatedly using the system until a draw does not occur. [Note that this is similar to the structure of W1(point-pairs) in bipoints.] Such systems can be represented by W1(SSi). This very natural conversion from two win/draw/loss systems to two win/loss systems produces scoring systems with expected durations equal to $\mu_i/(1-d_i)$ (i=1, 2). The probability player A wins under this converted system W1(SSi) is clearly equal to $p_i/(1-d_i)$ and the probability he loses is equal to $q_i/(1-d_i)$, and it follows from (4) above that the efficiency of W1(SS1) relative to W1(SS2) is equal to

$$\frac{((p_1 - q_1)/\mu_1)\ln(p_1/q_1)}{((p_2 - q_2)/\mu_2)\ln(p_2/q_2)} \qquad (5)$$

as the various (1-d_i) elements above cancel out in expression (4). Note that expression (5) applies to the situation in which draws are possible, whilst expression (4) is for the case in which draws are not possible. Also, note that the draw probabilities d_i are absent from expression (5).

Interestingly, this result is related to earlier work on the asymptotic efficiency of some (statistical) sequential probability ratio tests, SPRTs (or Wn systems with n large) which can be decomposed into small independent components called 'modules' (Pollard, 1990). These modules were equivalent to steps in a random walk, Z_i, which were independent variables on the integers …, -2, -1, 0, 1, 2, …. Using the approach of Cox and Miller (1965), the moment generating function of Z_i is defined by

$$f(\theta) = \sum_{j=-\infty}^{\infty} e^{-j\theta} P(Z_i = j),$$

and $\theta = 0$ is clearly one root of the equation $f(\theta) = 1$.

If $E(Z_i) \neq 0$, there is a unique real second root $\theta_0 \neq 0$ which has the same sign as $E(Z_i)$. (If $E(Z_i) = 0$, $\theta = 0$ is a double root). Pollard showed that the *asymptotic* efficiency of SPRT1 (with module 1) relative to SPRT2 (with module 2) is equal to

$$\frac{\theta_{0,1} E(Z_1)/ E(D_1)}{\theta_{0,2} E(Z_2)/ E(D_2)} \qquad (6)$$

where D_i is the expected duration of module i. Noting that for the scoring systems SS1 and SS2 under consideration in this section

$$E(D_i) = \mu_i,$$

$$E(Z_i) = p_i - q_i \text{ and}$$

$$f(\theta_i) = p_i e^{-\theta_i} + d_i + q_i e^{\theta_i}, \text{ and so we have}$$

$$\theta_{0,i} = \ln(p_i / q_i).$$

Thus, the asymptotic relative efficiency of these two quite general scoring systems (using the module approach and given by (6)) is identical to the non-asymptotic relative efficiency given by (5).

2.3. An Application of Win-Draw-Loss Structures to Tennis

'Game-pairs' with the win/draw/loss structure form an important 'building block' within tennis scoring systems. *In this next example we demonstrate how the efficiency of two alternative components within a scoring system can be directly compared without the need to assess the two full alternative systems in their entirety.*

Example 6

Here we consider the efficiency of a 'game-pair' using advantage tennis games relative to a 'game-pair' using '50-40' games (Pollard and Noble, 2004). In the '50-40' game, in order to win the game the server has to reach 50 (one more point than 40) before the receiver reaches 40. The receiver only needs to reach 40 in order to win the game.

Suppose player A has a point probability on service of 0.7, and player B has a point probability on service of 0.6. Using advantage games player A has a probability of 0.900788966 of winning a game on service, and the game has an expected duration of 5.831489655 points, whilst

player B has a probability of 0.735729231 of winning a game on service, and the game has an expected duration of 6.484184615 points. For '50-40' games these values are respectively 0.74431, 4.9579 points, 0.54432 and 4.9728 points.

Thus, for the advantage game-pair (p, d, q) is equal to (0.2380521928, 0.6889553495, 0.07299245775) and μ = 12.31567427. For the '50-40' game-pair (p, d, q) is equal to (0.3391671808, 0.5216556384, 0.1391771808) and μ =9.9307 points. Using (5) it follows that the efficiency of W1('50-40' game-pairs) relative to W1(advantage game-pairs) is equal to 1.132224066 when (pa, pb) = (0.7, 0.6). That is, '50-40' games are about 13% more efficient when (pa, pb) = (0.7, 0.6).

Not only is the '50-40' game-pair more efficient as a construct than is the advantage game-pair at (0.7, 0.6), but it has a smaller variance of duration. This is an attractive property of the '50-40' game since it is often the case that the more efficient of two systems has the disadvantage of having a larger variance of duration.

Thus, the '50-40' game is particularly relevant to men's doubles, as the point p-values for men's doubles average 0.65 or more.

Example 7

We finish this section with an example of using the methods in this paper to explain why the 'play-the-loser' (PL) service exchange mechanism is more efficient than 'play-the-winner' (PW) when service is an advantage, as in tennis. Using gpp to represent 'general point-pairs' (as in Pollard (1992)), consider the scoring system Wn(PLgpp). Here the match starts with an ab point-pair, a point-pair lost by player A is followed by the point-pair aa, a point-pair won by player A is followed by the point-pair bb, and a drawn point-pair is followed by the point-pair ab, and the match is won by the first player to be 2n points ahead. For this system we have, using an obvious notation,

$$\frac{P_{PL}}{Q_{PL}} = \frac{p_a q_b{}^{2n-1}}{p_b q_a{}^{2n-1}}$$

and

$$\frac{\mu_{PL}}{P_{PL} - Q_{PL}} = \frac{2(1 + (n-1)(p_a + p_b))}{p_a - p_b},$$

where p_a, p_b, q_a and q_b have been defined earlier.

For the associated system Wm(PWgpp), we have

$$\frac{\mu_{PW}}{P_{PW} - Q_{PW}} = \frac{2(1 + (m-1)(q_a + q_b))}{p_a - p_b}, \text{ and}$$

$$\frac{P_{PW}}{Q_{PW}} = \frac{p_a{}^{2m-1} q_b}{p_b{}^{2m-1} q_a}.$$

Now suppose we consider two such systems with

$$\frac{P_{PL}}{Q_{PL}} = \frac{P_{PW}}{Q_{PW}}.$$

It follows that

$(q_b / q_a)^{n-1} = (p_a / p_b)^{m-1}$, and hence, using the expansion for $\ln((1+x)/(1-x))$, we have

$$(\frac{(n-1)p}{(m-1)q})(\frac{1+(\delta/q)^2/3}{1+(\delta/p)^2/3}) = 1$$

where $p = (p_a + p_b)/2$, $q = 1 - p$ and $\delta = p_a - p$, and powers of δ^4 and higher are omitted. The second expression in brackets is greater than 1 in the tennis context ($p > 0.5$), and so the first expression must be less than 1. It follows that μ_{PL} is less than μ_{PW}, and so the PL system is the more efficient, as their P-values are equal. Correspondingly, the PW system is the more efficient when $p < 0.5$.

3. RESULTS

Suppose two players or two teams are playing a sport and there are two scoring systems (each with a win/loss outcome) under consideration for use, namely SS1 and SS2. Suppose SSi has an expected duration of μ_i points, the better player or team has a probability of p_i of winning under SSi (i = 1, 2), and a probability of q_i of losing (here $q_i = 1 - p_i$). It has been shown that *under these very general assumptions*, the efficiency of SS1 relative to SS2 is given by the ratio

$$\frac{((p_1 - q_1)/\mu_1)\ln(p_1/q_1)}{((p_2 - q_2)/\mu_2)\ln(p_2/q_2)}.$$

If the outcome under each scoring system SS1 and SS2 is instead win/draw/loss with probabilities $p_i/d_i/q_i$, then the efficiency of repeatedly playing SS1 until one player or team wins [namely W1(SS1)] relative to repeatedly playing SS2 until one player or team wins [namely W1(SS2)] is given by the same ratio.

It is clear that if SS1 is more efficient than SS2, and SS2 is more efficient than SS3, it follows that SS1 must be more efficient than SS3. Thus, the most efficient of a set of scoring systems can be identified.

4. CONCLUSIONS

Earlier work on the efficiency of scoring systems has been limited to those situations in which the underlying probability structures for the game being modelled had certain restrictive characteristics. Using those underlying characteristics it was possible to use interpolation methods to derive efficiency measures.

In this paper games that do not possess such restrictive probability structures have been considered, and it has been shown that extrapolation methods for deriving a relative efficiency measure can be developed and applied.

It turns out that that this extrapolation method can be used in many scoring system situations, and it produces exactly the same efficiency formula as that produced by the interpolated method. Thus, the method for measuring efficiency has been extended to a wider range of probabilistic situations.

The efficiency of nested scoring systems, whilst roughly multiplicative for the present tennis scoring system(s), has been shown to be exactly multiplicative for many situations.

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